15-819/18-879: Hybrid Systems Analysis & Theorem Proving

04: Tableau Provers in Propositional and First-order Logic

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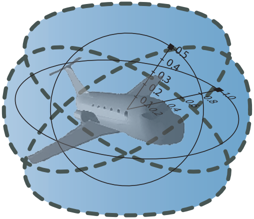

Outline

1. Propositional Logic
    - Tableau Proving
    - Soundness
    - Completeness

1. First-order Logic
    - Herbrand Theory
    - First-order Ground Tableaux
    - Free-Variable Tableaux
    - Soundness
    - Completeness

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Deciding/Analyzing Validity of PL_0

Proposition

Validity and satisfiability for propositional logic are decidable (by exhaustive enumeration) in exponential time.

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- Enumeration of interpretations does not scale

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Proposition

Validity and satisfiability for propositional logic are decidable (by exhaustive enumeration) in exponential time.

- Enumeration of interpretations does not scale
- Enumeration of interpretations is more tricky to generalize to FOL

Intuitive Approach

Example for an unsatisfiable PL_0 formula?

Intuitive Approach

Example for an unsatisfiable PL_0 formula?

$$(A \wedge B \rightarrow C) \wedge \neg(B \rightarrow C \vee \neg A)$$

Propositional Tableaux

"Systematically keep track of truth in tables"

$$\frac{A \wedge B}{} \qquad\qquad \frac{A \vee B}{}$$

Propositional Tableaux

"Systematically keep track of truth in tables"

$$\frac{A \land B}{\begin{array}{c}A\\B\end{array}} \qquad\qquad \frac{A \lor B}{}$$

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$$\frac{A \land B}{\begin{array}{c} A \\ B \end{array}} \qquad\qquad \frac{A \lor B}{A \quad B}$$

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$$\frac{A \land B}{\begin{array}{c}A\\B\end{array}} \qquad \frac{A \lor B}{A \quad B}$$

$$\frac{\neg(A \lor B)}{}$$

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$$\frac{\neg(A \to B)}{\begin{array}{c} A \\ \neg B \end{array}}$$

$$\frac{A \to B}{\neg A \quad B}$$

Propositional Tableaux

"Systematically keep track of truth in tables"

α-rules	β-rules	uniform
$\dfrac{A \land B}{\begin{array}{c}A\\ B\end{array}}$	$\dfrac{A \lor B}{A \quad B}$	$\dfrac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}}$
$\dfrac{\neg(A \lor B)}{\begin{array}{c}\neg A\\ \neg B\end{array}}$	$\dfrac{\neg(A \land B)}{\neg A \quad \neg B}$	$\dfrac{\beta}{\beta_1 \quad \beta_2}$
$\dfrac{\neg(A \rightarrow B)}{\begin{array}{c}A\\ \neg B\end{array}}$	$\dfrac{A \rightarrow B}{\neg A \quad B}$	

Propositional Tableaux

"Systematically keep track of truth in tables"

	α-rules	β-rules	uniform
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	$\dfrac{\neg(A \lor B)}{\begin{array}{c}\neg A\\\neg B\end{array}}$	$\dfrac{\neg(A \land B)}{\neg A \quad \neg B}$	$\dfrac{\beta}{\beta_1 \quad \beta_2}$
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	$\dfrac{\neg(A \lor B)}{\begin{array}{c}\neg A\\\neg B\end{array}}$	$\dfrac{\neg(A \land B)}{\neg A \quad \neg B}$	$\dfrac{\beta}{\beta_1 \quad \beta_2}$
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Propositional Tableaux

analytic tableaux with subformula property

	α-rules	β-rules	uniform
$\dfrac{\neg\neg A}{\begin{array}{c}A\\ A\end{array}}$	$\dfrac{A \land B}{\begin{array}{c}A\\ B\end{array}}$	$\dfrac{A \lor B}{A \quad B}$	$\dfrac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}}$
	$\dfrac{\neg(A \lor B)}{\begin{array}{c}\neg A\\ \neg B\end{array}}$	$\dfrac{\neg(A \land B)}{\neg A \quad \neg B}$	$\dfrac{\beta}{\beta_1 \quad \beta_2}$
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Tableau Extension

α-extension

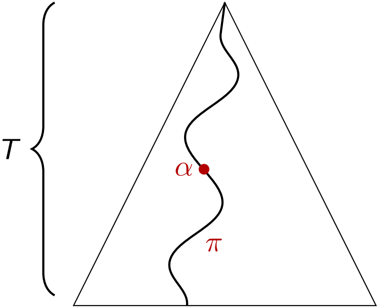

Tableau Extension

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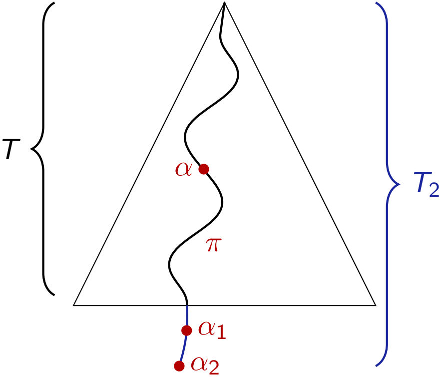
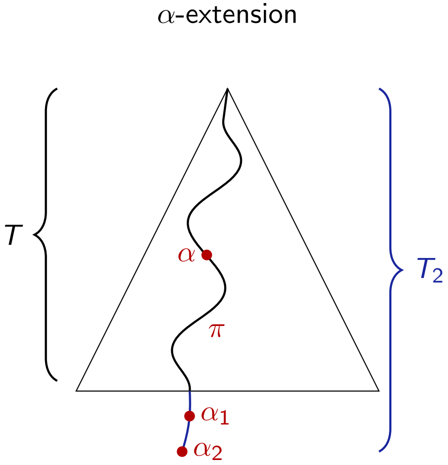


Tableau Extension



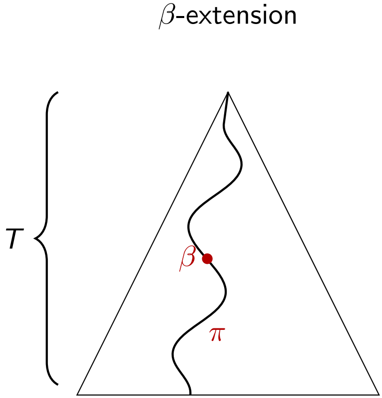
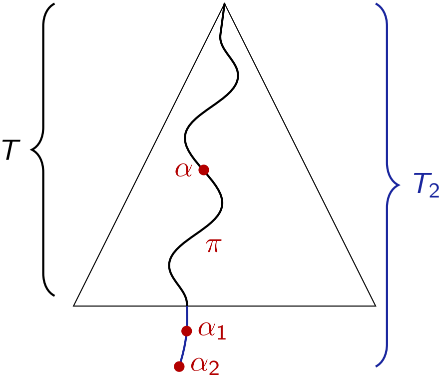
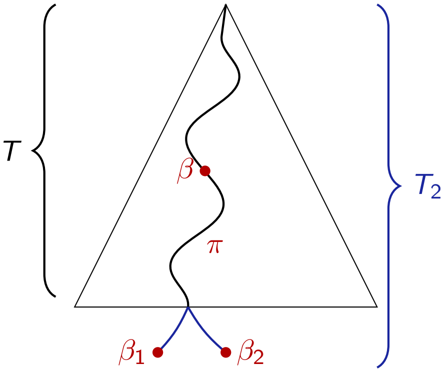


Tableau Extension

α-extension



β-extension



Propositional Tableaux - Formally

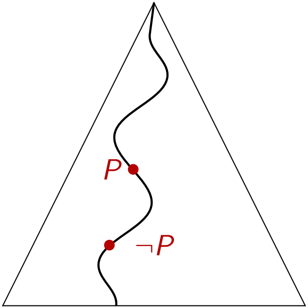
Definition (Tableaux)

A binary tree with PL_0 formulas as nodes is a *tableau* for formula F:

1. The tree consisting only of a node labelled $\neg F$ is a tableau.
2. Let G be the label of a node on some path π in a tableau T for F. If G is of the form of an α-premiss, then when extending π by two new nodes labelled with the α-conclusions, we obtain tableau T_2.
3. Let G be the label of a node on some path π in a tableau T for F. If G is of the form of a β-premiss, then when extending π by a branch to two new leaves labelled with the β-conclusions, we obtain a tableau T_2.

Tableau Path Closing

path closure



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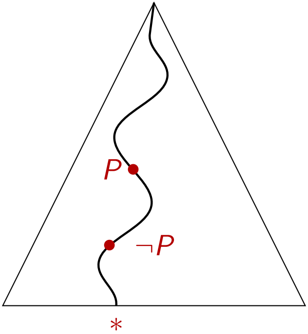


Provability

Definition (Provability)

- A path is *closed* iff it contains nodes labelled G and $\neg G$ for any formula G.
- A tableau is *closed* iff all its paths are closed.
- Formula F is *provable* (by propositional tableaux) iff there is a closed tableau for $\neg F$
  Notation: $\vdash_{\mathrm{tabPL_0}} F$

Tableau Proving Example

Example

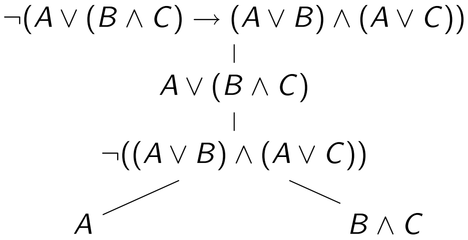
$A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$ is valid?

$$(A \lor (B \land C) \rightarrow (A \lor B) \land (A \lor C))$$

$$\neg(A \lor (B \land C) \to (A \lor B) \land (A \lor C))$$

$$\begin{array}{c}\neg(A \lor (B \land C) \to (A \lor B) \land (A \lor C)) \\ | \\ A \lor (B \land C)\end{array}$$

$$\begin{array}{c} \neg(A \lor (B \land C) \to (A \lor B) \land (A \lor C)) \\ | \\ A \lor (B \land C) \\ | \\ \neg((A \lor B) \land (A \lor C)) \end{array}$$



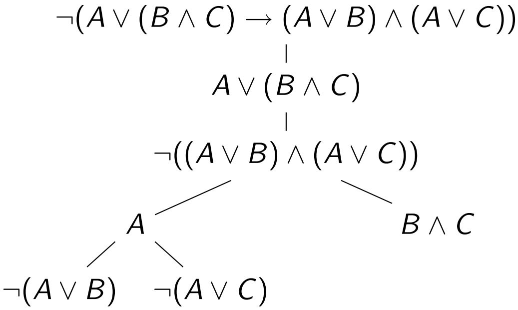
$\neg(A \lor (B \land C) \to (A \lor B) \land (A \lor C))$

$A \lor (B \land C)$

$\neg((A \lor B) \land (A \lor C))$

A

$B \land C$



$\neg(A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C))$

$A \vee (B \wedge C)$

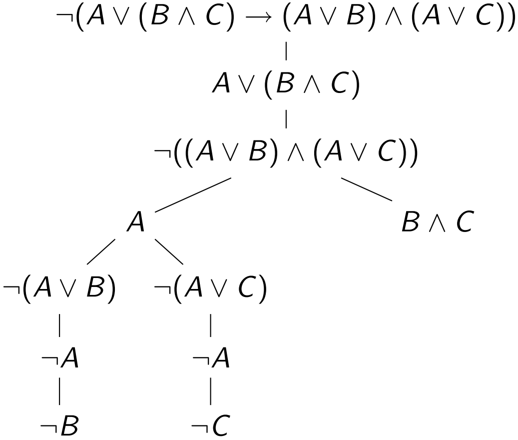
$\neg((A \vee B) \wedge (A \vee C))$

A

$B \wedge C$

$\neg(A \vee B)$

$\neg(A \vee C)$



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$\neg((A \lor B) \land (A \lor C))$

A

$B \land C$

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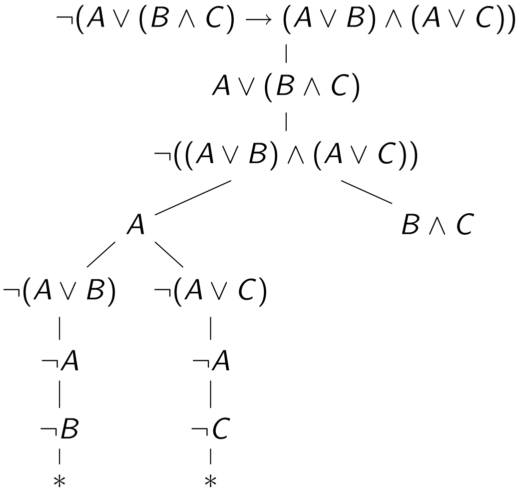
$\neg(A \lor C)$

$\neg A$

$\neg B$

$\neg A$

$\neg C$



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$A \lor (B \land C)$

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A

$B \land C$

$\neg(A \lor B)$

$\neg(A \lor C)$

$\neg A$

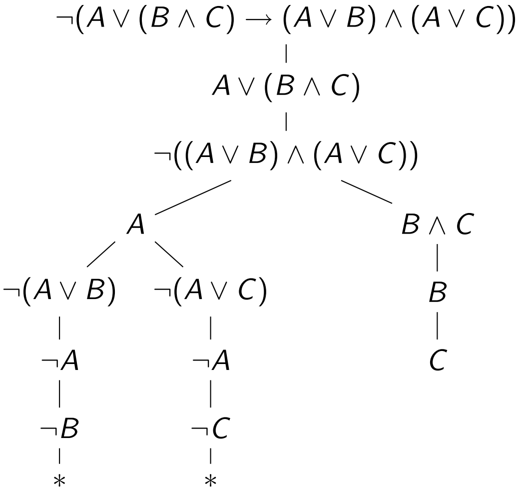
$\neg A$

$\neg B$

$\neg C$

$*$

$*$



$\neg(A \lor (B \land C) \rightarrow (A \lor B) \land (A \lor C))$

$A \lor (B \land C)$

$\neg((A \lor B) \land (A \lor C))$

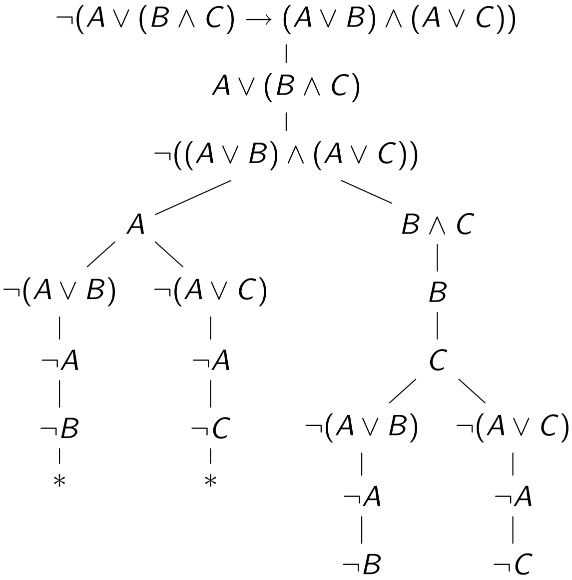
A $\quad$ $B \land C$

$\neg(A \lor B)$ $\quad$ $\neg(A \lor C)$ $\quad$ B

$\neg A$ $\quad$ $\neg A$ $\quad$ C

$\neg B$ $\quad$ $\neg C$

$*$ $\quad$ $*$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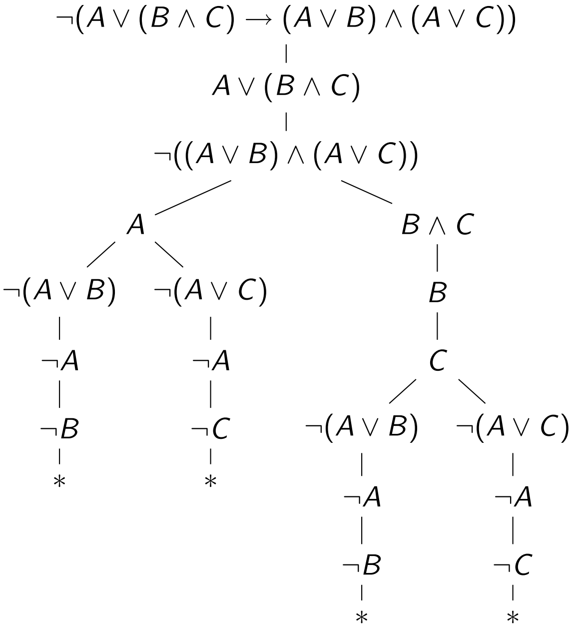

Soundness of PL_0 Tableaux

Theorem (Soundness)

All provable PL_0 *formulas* F *are valid:*

$$\vdash_{tab\mathrm{PL}_0} F \quad \Rightarrow \quad \vDash F$$

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Definition

An interpretation I is a *model* of tableau T iff there is a branch on which I satisfies all formulas.

Closed tableaux cannot have models.

Soundness of PL_0 Tableaux

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Proof.

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- Contrapositively, assume $\nvDash F$, then
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- Any tableau for F has a model: having a model transfers from each node to its children

▸ Check rules

□

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Proof.

- Contrapositively, assume $\nvDash F$, then
- $\neg F$ has a model
- Any tableau for F has a model: having a model transfers from each node to its children ▸ Check rules
- There is no closed tableau for F.

☐

Completeness of PL_0 Tableaux

Theorem (Completeness)

All valid PL_0 *formulas* F *are provable:*

$$\vDash F \quad \Rightarrow \quad \vdash_{tab\mathrm{PL}_0} F$$

Proof Sketch.

□

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Theorem (Completeness)

All valid PL_0 *formulas* F *are provable:*

$$\vDash F \quad \Rightarrow \quad \vdash_{tab\mathrm{PL}_0} F$$

Proof Sketch.

- For any exhausted open tableau for F there is a model of $\neg F$.

☐

Completeness of PL_0 Tableaux

Theorem (Completeness)

All valid PL_0 *formulas* F *are provable:*

$$\models F \quad \Rightarrow \quad \vdash_{tab\mathrm{PL}_0} F$$

Proof Sketch.

- For any exhausted open tableau for F there is a model of $\neg F$.
- Exhaustive iff every formula has been used on every open path.

☐

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First-order Logic

What is the most naïve syntactic interpretation?

Herbrand Model

Definition (Herbrand Model)

An interpretation with free semantics for terms is called *Herbrand model*:

1. Ground terms $Term^0(\Sigma)$ over Σ as domain D (i.e. no variables)
2. $I(f) : D^n \rightarrow D; (t_1, \dots, t_n) \mapsto f(t_1, \dots, t_n)$ for each function symbol f of arity n

Let Γ be a set of closed universal formulas. $Term^0(\Sigma)(\Gamma)$ is the set of all ground term instances of the formulas in Γ, i.e., with ground terms instantiated for the variables of the universal quantifier prefix.

Herbrand's Theorem

Theorem (Herbrand)

Let Γ be a (suitable) set of first-order formulas (i.e. closed universal formulas without equality with Σ having at least one constant).

$$\begin{aligned} \Gamma \text{ has a model } &\iff \Gamma \text{ has a Herbrand model, i.e. free on ground terms} \\ &\iff \text{ground terms } Term^0(\Sigma)(\Gamma) \text{ of } \Gamma \text{ have a model} \end{aligned}$$

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Theorem (Finiteness Corollary to Compactness)

Γ has a model iff every finite subset of Γ has.

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Validity of FOL *is semidecidable.*

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- F valid $\iff$ $\neg F$ unsatisfiable

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- $\iff$ $Term^0(\Sigma)(\neg F)$ have no model

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Γ *has a model* $\iff$ Γ *has a Herbrand model, i.e. free on ground terms*

$\iff$ *ground terms* $Term^0(\Sigma)(\Gamma)$ *of* Γ *have a model*

Theorem (Finiteness Corollary to Compactness)

Γ *has a model iff every finite subset of* Γ *has.*

Corollary

Validity of FOL *is semidecidable.*

- F valid $\iff$ $\neg F$ unsatisfiable
- $\iff$ $Term^0(\Sigma)(\neg F)$ have no model
- $\iff$ some finite subet of $Term^0(\Sigma)(\neg F)$ has no Herbrand model

Arbitrary Formulas versus Universal Formulas

- $\forall x\, \forall y\, (p(x, y) \rightarrow p(y, x))$ is universal

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- $\forall x\, \forall y\, (p(x, y) \rightarrow p(y, x))$ is universal
- $\exists x\, \forall y\, (p(x, y) \rightarrow p(y, x))$ is not universal

Arbitrary Formulas versus Universal Formulas

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- Name existentials: $\forall y\, (p(x, a) \rightarrow p(a, x))$ is universal (with new constant symbol a)

Arbitrary Formulas versus Universal Formulas

- $\forall x\, \forall y\, (p(x, y) \rightarrow p(y, x))$ is universal
- $\exists x\, \forall y\, (p(x, y) \rightarrow p(y, x))$ is not universal
- Name existentials: $\forall y\, (p(x, a) \rightarrow p(a, x))$ is universal (with new constant symbol a)
- Are these formulas equivalent?

Arbitrary Formulas versus Universal Formulas

- $\forall x\, \forall y\, (p(x,y) \rightarrow p(y,x))$ is universal
- $\exists x\, \forall y\, (p(x,y) \rightarrow p(y,x))$ is not universal
- Name existentials: $\forall y\, (p(x,a) \rightarrow p(a,x))$ is universal (with new constant symbol a)
- Are these formulas equivalent? No! But satisfiability-equivalent!

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- $\forall y\, (p(s(y), y) \rightarrow p(y, s(y)))$ is universal (with new Skolem function s)
- Both are satisfiability-equivalent
- In group theory, $\forall x\, \exists y\, x \cdot y = 1$ that inverse would be called $i(x)$, obviously depending on x: $\forall x\, x \cdot i(x) = 1$

Skolemization and Satisfiability-Equivalence

Lemma

For every FOL *formula there is a universal formula that is satisfiability-equivalent.*

Skolemization and Satisfiability-Equivalence

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Proof.

Iterative Skolemization

□

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Iterative Skolemization

$$\forall x \exists y \left(p(x,y) \to \forall z\, \exists u\, (q(x,u,z) \land p(u,y))\right)$$

☐

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$$\forall x \exists y \left(p(x,y) \rightarrow \forall z\, \exists u \left(q(x,u,z) \land p(u,y)\right)\right)$$

☐

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Proof.

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$$\forall x\,\big(p(x, s(x)) \to \forall z\, \exists u(q(x, u, z) \land p(u, s(x)))\big)$$

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$$\forall x\,\big(p(x,s(x)) \rightarrow \forall z\,(q(x,t(x,z),z) \land p(t(x,z),s(x)))\big)$$

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$$\forall x \exists y \left(p(x, y) \to \forall z\, \exists u (q(x, u, z) \land p(u, y))\right) \to \exists v\, q(v, x, y)$$

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$$\forall x\,(p(x,s(x)) \rightarrow \forall z\,(q(x,t(x,z),z) \land p(t(x,z),s(x)))) \rightarrow q(f(x,y),x,y)$$

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Nonsense! Existential quantifiers in negative positions are more like universals! ☐

Skolemization and Satisfiability-Equivalence

Lemma

For every FOL *formula there is a universal formula that is satisfiability-equivalent, and it can be constructed effectively.*

Proof.

Prenex normal form + iterative Skolemization

☐

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- $F \land \forall x\, G \equiv$

$\square$

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- $F \land \forall x\, G \equiv \forall x\, (F \land G)$ if $x \notin FV(F)$

☐

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☐

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- $(\exists x\, F) \rightarrow G \equiv$

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- $(\forall x\, F) \rightarrow G \equiv \exists x\, (F \rightarrow G)$ if $x \notin FV(G)$

□

First-order Tableaux

α-rules	β-rules
$$\frac{A \land B}{\begin{array}{c}A\\ B\end{array}}$$	$$\frac{A \lor B}{A \quad B}$$

First-order Tableaux

α-rules

$$\frac{A \land B}{\begin{array}{c} A \\ B \end{array}}$$

β-rules

$$\frac{A \lor B}{A \quad B}$$

$$\frac{\forall x\, A(x)}{}$$

First-order Tableaux

α-rules

$$\frac{A \wedge B}{\begin{array}{c}A\\B\end{array}}$$

β-rules

$$\frac{A \vee B}{A \quad B}$$

$$\frac{\forall x\, A(x)}{A(t)}$$

First-order Ground Tableaux

α-rules

$$\frac{A \wedge B}{\begin{array}{c}A\\B\end{array}}$$

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$$\frac{\forall x\, A(x)}{A(t)}$$

t ground

First-order Ground Tableaux

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t ground

c new constant

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γ^*-rules

$$\frac{\forall x\, A(x)}{A(t)}$$

t ground

δ-rules

$$\frac{\exists x\, A(x)}{A(c)}$$

c new constant

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$$\frac{A \wedge B}{\begin{array}{c} A \\ B \end{array}}$$

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$$\frac{\forall x\, A(x)}{A(t)}$$

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t ground

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$$\frac{\neg\exists x\, A(x)}{}$$

First-order Ground Tableaux

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$$\frac{A \land B}{\begin{array}{c}A\\B\end{array}}$$

β-rules

$$\frac{A \lor B}{A \quad B}$$

γ^*-rules

$$\frac{\forall x\, A(x)}{A(t)}$$

t ground

$$\frac{\neg\exists x\, A(x)}{\neg A(t)}$$

δ-rules

$$\frac{\exists x\, A(x)}{A(c)}$$

c new constant

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Tableau Extension

α-extension

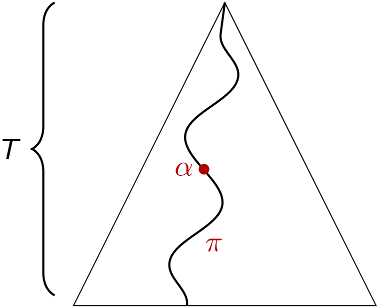


Tableau Extension

α-extension

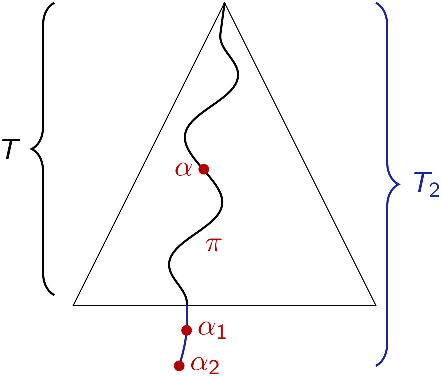
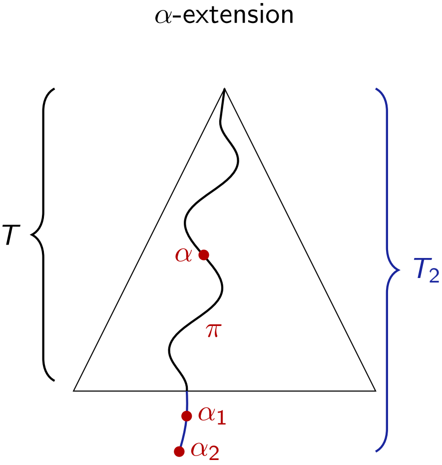


Tableau Extension



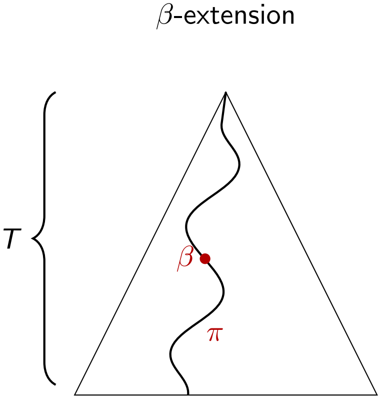
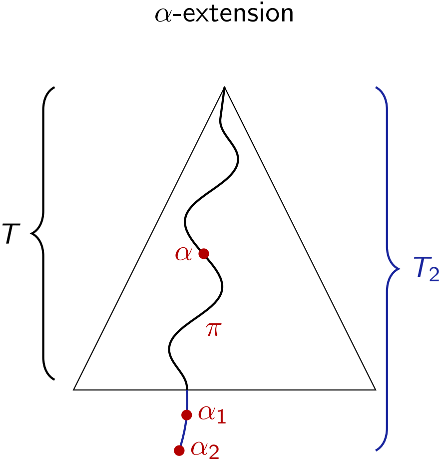


Tableau Extension



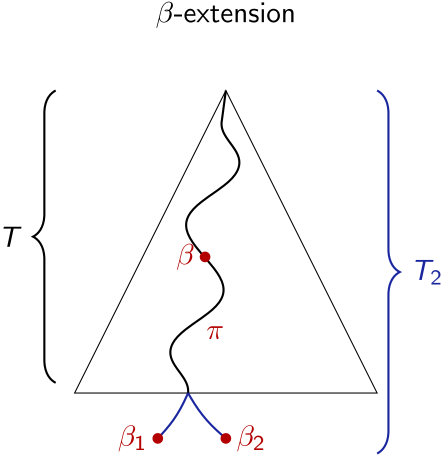


Tableau Extension

γ^*-extension

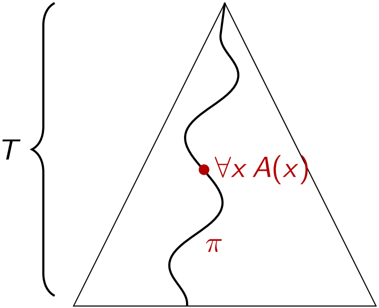


Tableau Extension

γ^*-extension

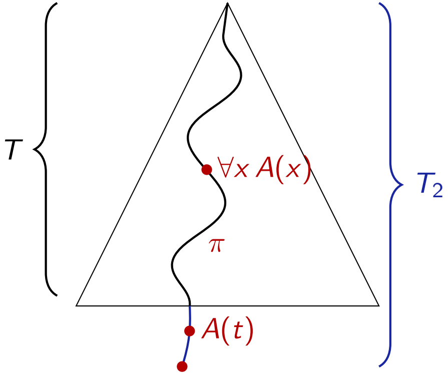


Tableau Extension

γ^*-extension

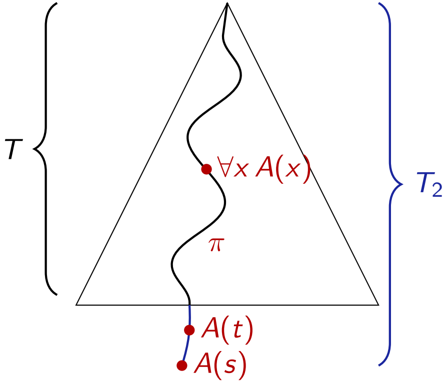
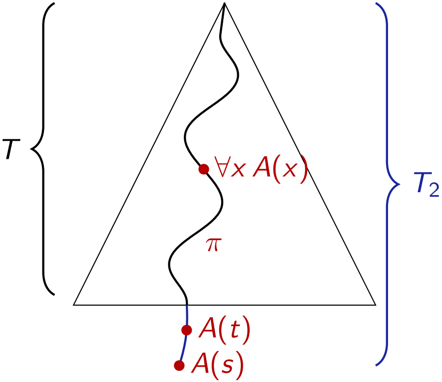
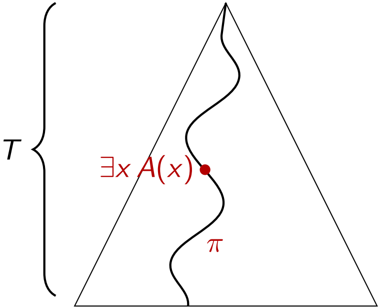


Tableau Extension

γ^*-extension

δ-extension





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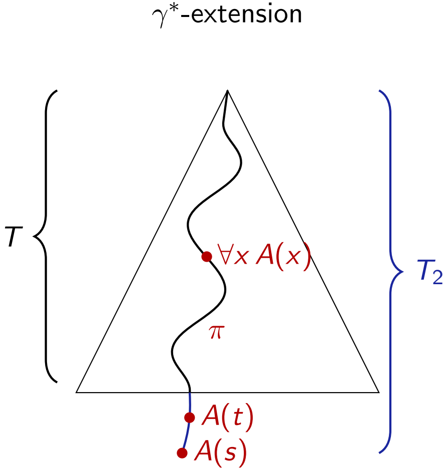


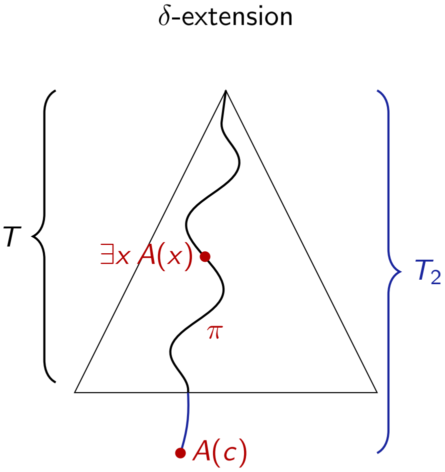
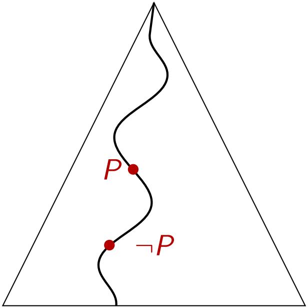


Tableau Path Closing

path closure



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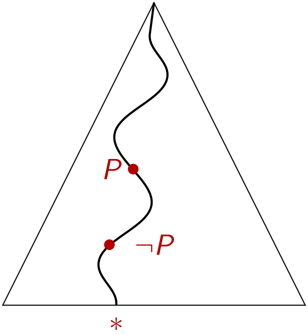

Ground Tableaux Example

$$(\exists y\, \forall x\, \mathrm{ctrl}(x, y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x, y))$$

Ground Tableaux Example

$$\neg(\exists y\, \forall x\, \mathrm{ctrl}(x, y) \to \forall x\, \exists y\, \mathrm{ctrl}(x, y))$$

Ground Tableaux Example

$$\begin{array}{c} \neg(\exists y\,\forall x\,\mathrm{ctrl}(x, y) \rightarrow \forall x\,\exists y\,\mathrm{ctrl}(x, y)) \\ | \\ \exists y\,\forall x\,\mathrm{ctrl}(x, y) \end{array}$$

Ground Tableaux Example

$$\begin{array}{c}
\neg(\exists y\, \forall x\, \mathrm{ctrl}(x,y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x,y)) \\
| \\
\exists y\, \forall x\, \mathrm{ctrl}(x,y) \\
| \\
\neg\forall x\, \exists y\, \mathrm{ctrl}(x,y)
\end{array}$$

Ground Tableaux Example

$$\neg(\exists y\, \forall x\, \mathrm{ctrl}(x, y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x, y))$$

$$|$$

$$\exists y\, \forall x\, \mathrm{ctrl}(x, y)$$

$$|$$

$$\neg\forall x\, \exists y\, \mathrm{ctrl}(x, y)$$

$$|$$

$$\forall x\, \mathrm{ctrl}(x, c)$$

Ground Tableaux Example

$$\begin{array}{c}
\neg(\exists y\, \forall x\, \mathrm{ctrl}(x, y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x, y)) \\
| \\
\exists y\, \forall x\, \mathrm{ctrl}(x, y) \\
| \\
\neg\forall x\, \exists y\, \mathrm{ctrl}(x, y) \\
| \\
\forall x\, \mathrm{ctrl}(x, c) \\
| \\
\neg\exists y\, \mathrm{ctrl}(d, y)
\end{array}$$

Ground Tableaux Example

$$\begin{array}{c}
\neg(\exists y\,\forall x\,\mathrm{ctrl}(x,y) \rightarrow \forall x\,\exists y\,\mathrm{ctrl}(x,y)) \\
| \\
\exists y\,\forall x\,\mathrm{ctrl}(x,y) \\
| \\
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| \\
\forall x\,\mathrm{ctrl}(x,c) \\
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$$|$$

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$$|$$

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$$|$$

$$\forall x\, \mathrm{ctrl}(x, c)$$

$$|$$

$$\neg\exists y\, \mathrm{ctrl}(d, y)$$

$$|$$

$$\neg\mathrm{ctrl}(d, c)$$

$$|$$

$$\mathrm{ctrl}(d, c)$$

$$|$$

$$*$$

Ground Tableaux Example

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$$|$$

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$$|$$

$$\forall x\, \mathrm{ctrl}(x, c)$$

$$|$$

$$\neg\exists y\, \mathrm{ctrl}(d, y)$$

$$|$$

$$\neg\mathrm{ctrl}(d, c)$$

$$|$$

$$\mathrm{ctrl}(d, c)$$

$$|$$

$$*$$

What is the difficulty?

Ground Tableaux Example

$$\begin{array}{c}
\neg(\exists y\, \forall x\, \mathrm{ctrl}(x,y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x,y)) \\
| \\
\exists y\, \forall x\, \mathrm{ctrl}(x,y) \\
| \\
\neg\forall x\, \exists y\, \mathrm{ctrl}(x,y) \\
| \\
\forall x\, \mathrm{ctrl}(x,c) \\
| \\
\neg\exists y\, \mathrm{ctrl}(d,y) \\
| \\
\neg\mathrm{ctrl}(d,c) \\
| \\
\mathrm{ctrl}(d,c) \\
| \\
*
\end{array}$$

What is the difficulty?

Fighting Eager Instantiations

Fighting Eager Instantiations

Idea: delay instantiation

Lazy Instantiation Tableaux Motivation

$$(\exists y\, \forall x\, \mathrm{ctrl}(x, y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x, y))$$

Lazy Instantiation Tableaux Motivation

$$\neg(\exists y\,\forall x\,\mathrm{ctrl}(x,y) \rightarrow \forall x\,\exists y\,\mathrm{ctrl}(x,y))$$

Lazy Instantiation Tableaux Motivation

$$\begin{array}{c}
\neg(\exists y\, \forall x\, \mathrm{ctrl}(x,y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x,y)) \\
| \\
\exists y\, \forall x\, \mathrm{ctrl}(x,y)
\end{array}$$

Lazy Instantiation Tableaux Motivation

$$\begin{array}{c}
\neg(\exists y\,\forall x\,\mathrm{ctrl}(x,y) \rightarrow \forall x\,\exists y\,\mathrm{ctrl}(x,y)) \\
| \\
\exists y\,\forall x\,\mathrm{ctrl}(x,y) \\
| \\
\neg\forall x\,\exists y\,\mathrm{ctrl}(x,y)
\end{array}$$

Lazy Instantiation Tableaux Motivation

$$\neg(\exists y\,\forall x\,\mathrm{ctrl}(x,y) \rightarrow \forall x\,\exists y\,\mathrm{ctrl}(x,y))$$

$$|$$

$$\exists y\,\forall x\,\mathrm{ctrl}(x,y)$$

$$|$$

$$\neg\forall x\,\exists y\,\mathrm{ctrl}(x,y)$$

$$|$$

$$\forall x\,\mathrm{ctrl}(x,c)$$

Lazy Instantiation Tableaux Motivation

$$\begin{array}{c}
\neg(\exists y\, \forall x\, \mathrm{ctrl}(x,y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x,y)) \\
| \\
\exists y\, \forall x\, \mathrm{ctrl}(x,y) \\
| \\
\neg \forall x\, \exists y\, \mathrm{ctrl}(x,y) \\
| \\
\forall x\, \mathrm{ctrl}(x,c) \\
| \\
\neg \exists y\, \mathrm{ctrl}(d,y)
\end{array}$$

Lazy Instantiation Tableaux Motivation

$$\begin{array}{c}
\neg(\exists y\, \forall x\, \mathrm{ctrl}(x, y) \rightarrow \forall x\, \exists y\, \mathrm{ctrl}(x, y)) \\
| \\
\exists y\, \forall x\, \mathrm{ctrl}(x, y) \\
| \\
\neg \forall x\, \exists y\, \mathrm{ctrl}(x, y) \\
| \\
\forall x\, \mathrm{ctrl}(x, c) \\
| \\
\neg \exists y\, \mathrm{ctrl}(d, y) \\
| \\
\neg \mathrm{ctrl}(d, Y)
\end{array}$$

Lazy Instantiation Tableaux Motivation

$$\begin{array}{c}
\neg(\exists y\,\forall x\,\mathrm{ctrl}(x,y) \rightarrow \forall x\,\exists y\,\mathrm{ctrl}(x,y)) \\
| \\
\exists y\,\forall x\,\mathrm{ctrl}(x,y) \\
| \\
\neg\forall x\,\exists y\,\mathrm{ctrl}(x,y) \\
| \\
\forall x\,\mathrm{ctrl}(x,c) \\
| \\
\neg\exists y\,\mathrm{ctrl}(d,y) \\
| \\
\neg\mathrm{ctrl}(d,Y) \\
| \\
\mathrm{ctrl}(X,c)
\end{array}$$

Lazy Instantiation Tableaux Motivation

$$\begin{array}{c}
\neg(\exists y\,\forall x\,\mathrm{ctrl}(x,y) \rightarrow \forall x\,\exists y\,\mathrm{ctrl}(x,y)) \\
| \\
\exists y\,\forall x\,\mathrm{ctrl}(x,y) \\
| \\
\neg\forall x\,\exists y\,\mathrm{ctrl}(x,y) \\
| \\
\forall x\,\mathrm{ctrl}(x,c) \\
| \\
\neg\exists y\,\mathrm{ctrl}(d,y) \\
| \\
\neg\mathrm{ctrl}(d,Y) \\
| \\
\mathrm{ctrl}(X,c) \\
| \\
* \\
\{d/X\}\{c/Y\}
\end{array}$$

First-order Free-Variable Tableaux

α-rules

$$\frac{A \wedge B}{\begin{array}{c} A \\ B \end{array}}$$

β-rules

$$\frac{A \vee B}{A \quad B}$$

First-order Free-Variable Tableaux

α-rules

$$\frac{A \land B}{\begin{array}{c}A\\B\end{array}}$$

β-rules

$$\frac{A \lor B}{A \quad B}$$

$$\frac{\forall x\, A(x)}{}$$

First-order Free-Variable Tableaux

α-rules

$$\frac{A \land B}{\begin{array}{c}A\\B\end{array}}$$

β-rules

$$\frac{A \lor B}{A \quad B}$$

$$\frac{\forall x\, A(x)}{A(X)}$$

First-order Free-Variable Tableaux

α-rules

$$\frac{A \land B}{\begin{array}{c} A \\ B \end{array}}$$

β-rules

$$\frac{A \lor B}{A \quad B}$$

$$\frac{\forall x\, A(x)}{A(X)}$$

X new variable on path

First-order Free-Variable Tableaux

α-rules

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$$\frac{\exists x\, A(x)}{}$$

X new variable on path

First-order Free-Variable Tableaux

α-rules

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β-rules

$$\frac{A \lor B}{A \quad B}$$

$$\frac{\forall x\, A(x)}{A(X)}$$

$$\frac{\exists x\, A(x)}{A(s(X_1, \dots, X_n))}$$

X new variable on path

First-order Free-Variable Tableaux

α-rules

$$\frac{A \land B}{\begin{array}{c}A\\B\end{array}}$$

β-rules

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X new variable on path

$FV(\exists x\, A(x)) = \{X_1, \ldots, X_n\}$, s new

First-order Free-Variable Tableaux

α-rules

$$\frac{A \land B}{\begin{array}{c}A\\B\end{array}}$$

β-rules

$$\frac{A \lor B}{A \quad B}$$

γ^*-rules

$$\frac{\forall x\, A(x)}{A(X)}$$

δ-rules

$$\frac{\exists x\, A(x)}{A(s(X_1,\dots,X_n))}$$

X new variable on path

$FV(\exists x\, A(x)) = \{X_1,\dots,X_n\}$, s new

First-order Free-Variable Tableaux

α-rules

$$\frac{A \land B}{\begin{matrix} A \\ B \end{matrix}}$$

β-rules

$$\frac{A \lor B}{A \quad B}$$

γ^*-rules

$$\frac{\forall x\, A(x)}{A(X)}$$

X new variable on path

$$\frac{\neg\exists x\, A(x)}{}$$

δ-rules

$$\frac{\exists x\, A(x)}{A(s(X_1, \dots, X_n))}$$

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First-order Free-Variable Tableaux

α-rules

$$\frac{A \land B}{\begin{array}{c} A \\ B \end{array}}$$

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γ^*-rules

$$\frac{\forall x\, A(x)}{A(X)}$$

δ-rules

$$\frac{\exists x\, A(x)}{A(s(X_1, \dots, X_n))}$$

X new variable on path

$$\frac{\neg \exists x\, A(x)}{\neg A(X)}$$

$FV(\exists x\, A(x)) = \{X_1, \dots, X_n\}$, s new

First-order Free-Variable Tableaux

α-rules

$$\frac{A \land B}{\begin{array}{c} A \\ B \end{array}}$$

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First-order Free-Variable Tableaux

α-rules

$$\frac{A \land B}{\begin{matrix} A \\ B \end{matrix}}$$

β-rules

$$\frac{A \lor B}{A \quad B}$$

γ^*-rules

$$\frac{\forall x\, A(x)}{A(X)}$$

X new variable on path

$$\frac{\neg \exists x\, A(x)}{\neg A(X)}$$

δ-rules

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$FV(\exists x\, A(x)) = \{X_1, \ldots, X_n\}$, s new

$$\frac{\neg \forall x\, A(x)}{\neg A(s(X_1, \ldots, X_n))}$$

Tableau Extension

γ^*-extension

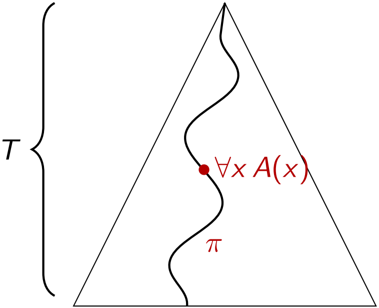


Tableau Extension

γ^*-extension

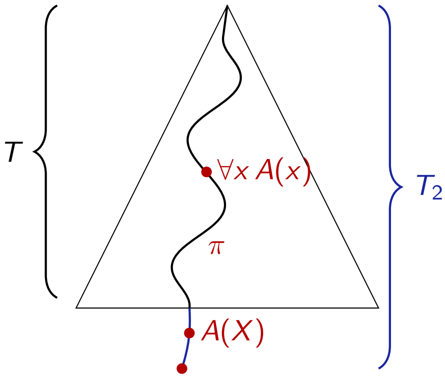


Tableau Extension

γ^*-extension

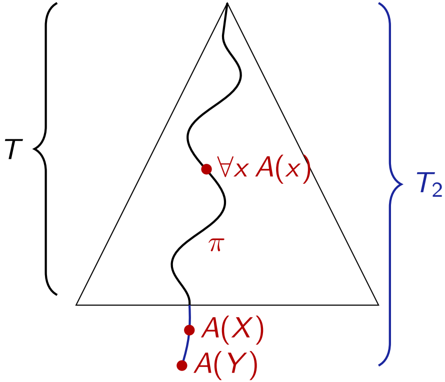
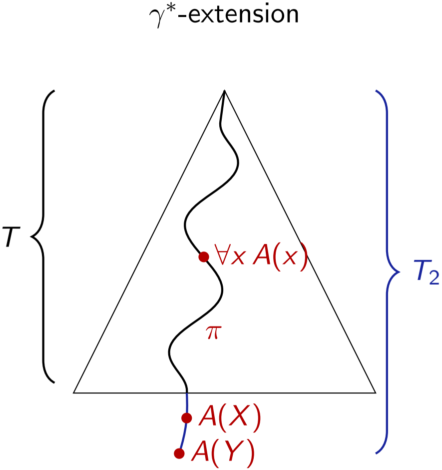


Tableau Extension



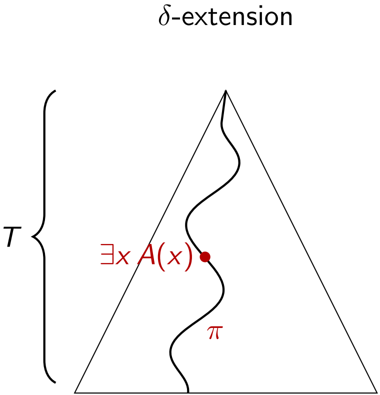
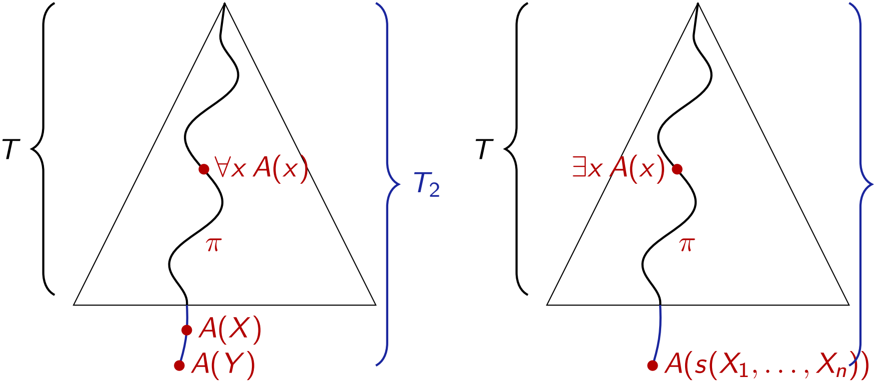


Tableau Extension

γ^*-extension

δ-extension



Free-Variable Tableaux Closing

$$\frac{\begin{array}{c}\neg\phi(t_1,\dots,t_n)\\ \vdots\\ \psi(s_1,\dots,s_n)\end{array}}{*\sigma}$$

$$\sigma(\phi(t_1,\dots,t_n)) = \sigma(\psi(t_1,\dots,t_n))$$

Free-Variable Tableaux Closing

$$\frac{\begin{array}{c}\neg\phi(t_1,\dots,t_n)\\ \vdots\\ \psi(s_1,\dots,s_n)\end{array}}{*\sigma}$$

$$\sigma(\phi(t_1,\dots,t_n)) = \sigma(\psi(t_1,\dots,t_n))$$

With substitution σ applied to full tableau for closing

Tableau Path Closing

path closure

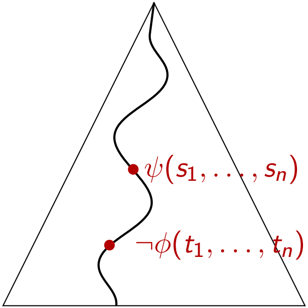
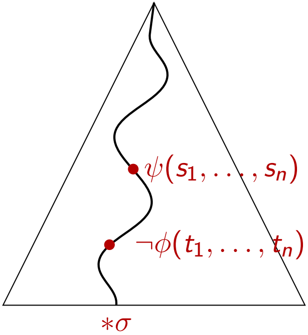


Tableau Path Closing

path closure



Provability

Definition (Provability)

- A path can be *closed* by substitution σ iff it contains nodes labelled G and $\neg H$ for any formulas G, H with $\sigma G = \sigma H$. When closing this path, σ is applied to the full tableaux.
- A tableau is *closed* iff all its paths are closed.
- Formula F is *provable* (by first-order tableaux) iff there is a closed tableau for $\neg F$
  Notation: $\vdash_{\text{tabFOL}} F$

Variation

Definition (Substitution)

A total endomorphism $\sigma : \mathrm{Trm}(\Sigma) \to \mathrm{Trm}(\Sigma)$ of finite support in V:

$$\{x \in V \ : \ \sigma x \neq x\} \text{ is finite and } \sigma = \mathrm{id} \text{ on } \Sigma$$

$$\sigma(f(t_1, \dots, t_n)) = \sigma f(\sigma t_1, \dots, \sigma t_n)$$

$$\sigma(p(t_1, \dots, t_n)) = \sigma p(\sigma t_1, \dots, \sigma t_n)$$

$$\sigma \neg A = \neg \sigma A$$

$$\sigma A \wedge B = \sigma A \wedge \sigma B \quad \text{(likewise for } \vee, \rightarrow, \leftrightarrow\text{)}$$

$$\sigma \forall x\, A = \forall x\, \sigma_x A \ \text{ where } \sigma_x \text{ is like } \sigma \text{ except that } \sigma_x(x) = x$$

$$\sigma \exists x\, A = \exists x\, \sigma_x A \ \text{ where } \sigma_x \text{ is like } \sigma \text{ except that } \sigma_x(x) = x$$

"simultaneously replace all free occurrences of any variable x by σx"

Variation

σ only admissible for formula F iff no replaced variable x occurs within scope of a quantifier for σx

Definition (Substitution)

A total endomorphism $\sigma : \mathrm{Trm}(\Sigma) \to \mathrm{Trm}(\Sigma)$ of finite support in V:

$$\{x \in V \;:\; \sigma x \neq x\} \text{ is finite and } \sigma = \mathrm{id} \text{ on } \Sigma$$

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"simultaneously replace all free occurrences of any variable x by σx"

Free Variable Tableaux Example

$$(\forall x\, \exists y\, \mathrm{ctrl}(x, y) \rightarrow \exists y\, \forall x\, \mathrm{ctrl}(x, y))$$

Free Variable Tableaux Example

$$\neg(\forall x\, \exists y\, \mathrm{ctrl}(x, y) \rightarrow \exists y\, \forall x\, \mathrm{ctrl}(x, y))$$

Free Variable Tableaux Example

$$\begin{array}{c}
\neg(\forall x\, \exists y\, \mathrm{ctrl}(x,y) \rightarrow \exists y\, \forall x\, \mathrm{ctrl}(x,y)) \\
| \\
\forall x\, \exists y\, \mathrm{ctrl}(x,y)
\end{array}$$

Free Variable Tableaux Example

$$\begin{array}{c} \neg(\forall x\, \exists y\, \mathrm{ctrl}(x,y) \rightarrow \exists y\, \forall x\, \mathrm{ctrl}(x,y)) \\ | \\ \forall x\, \exists y\, \mathrm{ctrl}(x,y) \\ | \\ \neg\exists y\, \forall x\, \mathrm{ctrl}(x,y) \end{array}$$

Free Variable Tableaux Example

$$\begin{array}{c}
\neg(\forall x\, \exists y\, \mathrm{ctrl}(x, y) \rightarrow \exists y\, \forall x\, \mathrm{ctrl}(x, y)) \\
| \\
\forall x\, \exists y\, \mathrm{ctrl}(x, y) \\
| \\
\neg\exists y\, \forall x\, \mathrm{ctrl}(x, y) \\
| \\
\exists y\, \mathrm{ctrl}(X, y)
\end{array}$$

Free Variable Tableaux Example

$$\begin{array}{c}
\neg(\forall x\, \exists y\, \mathrm{ctrl}(x, y) \rightarrow \exists y\, \forall x\, \mathrm{ctrl}(x, y)) \\
| \\
\forall x\, \exists y\, \mathrm{ctrl}(x, y) \\
| \\
\neg\exists y\, \forall x\, \mathrm{ctrl}(x, y) \\
| \\
\exists y\, \mathrm{ctrl}(X, y) \\
| \\
\mathrm{ctrl}(X, c(X))
\end{array}$$

Free Variable Tableaux Example

$$\begin{array}{c}
\neg(\forall x\,\exists y\,\mathrm{ctrl}(x,y) \rightarrow \exists y\,\forall x\,\mathrm{ctrl}(x,y)) \\
| \\
\forall x\,\exists y\,\mathrm{ctrl}(x,y) \\
| \\
\neg\exists y\,\forall x\,\mathrm{ctrl}(x,y) \\
| \\
\exists y\,\mathrm{ctrl}(X,y) \\
| \\
\mathrm{ctrl}(X,c(X)) \\
| \\
\neg\forall x\,\mathrm{ctrl}(x,Y)
\end{array}$$

Free Variable Tableaux Example

$$
\begin{array}{c}
\neg(\forall x\,\exists y\,\mathrm{ctrl}(x,y) \rightarrow \exists y\,\forall x\,\mathrm{ctrl}(x,y)) \\
| \\
\forall x\,\exists y\,\mathrm{ctrl}(x,y) \\
| \\
\neg\exists y\,\forall x\,\mathrm{ctrl}(x,y) \\
| \\
\exists y\,\mathrm{ctrl}(X,y) \\
| \\
\mathrm{ctrl}(X,c(X)) \\
| \\
\neg\forall x\,\mathrm{ctrl}(x,Y) \\
| \\
\neg\mathrm{ctrl}(d(Y),Y)
\end{array}
$$

Free Variable Tableaux Example

$$\neg(\forall x\, \exists y\, \mathrm{ctrl}(x, y) \rightarrow \exists y\, \forall x\, \mathrm{ctrl}(x, y))$$

$$|$$

$$\forall x\, \exists y\, \mathrm{ctrl}(x, y)$$

$$|$$

$$\neg\exists y\, \forall x\, \mathrm{ctrl}(x, y)$$

$$|$$

$$\exists y\, \mathrm{ctrl}(X, y)$$

$$|$$

$$\mathrm{ctrl}(X, c(X))$$

$$|$$

$$\neg\forall x\, \mathrm{ctrl}(x, Y)$$

$$|$$

$$\neg\mathrm{ctrl}(d(Y), Y)$$

$$|$$

$$*$$

$$\{d(Y)/X\}\{c(X)/Y\}$$

Free Variable Tableaux Example

$$\neg(\forall x\,\exists y\,\mathrm{ctrl}(x,y) \rightarrow \exists y\,\forall x\,\mathrm{ctrl}(x,y))$$

$$|$$

$$\forall x\,\exists y\,\mathrm{ctrl}(x,y)$$

$$|$$

$$\neg\exists y\,\forall x\,\mathrm{ctrl}(x,y)$$

$$|$$

$$\exists y\,\mathrm{ctrl}(X,y)$$

$$|$$

$$\mathrm{ctrl}(X,c(X))$$

$$|$$

$$\neg\forall x\,\mathrm{ctrl}(x,Y)$$

$$|$$

$$\neg\mathrm{ctrl}(d(Y),Y)$$

$$|$$

$$*$$

$$\{d(Y)/X\}\{c(d(Y))/Y\}$$

Free Variable Tableaux Example

$$\begin{array}{c}
\neg(\forall x\,\exists y\,\mathrm{ctrl}(x,y) \rightarrow \exists y\,\forall x\,\mathrm{ctrl}(x,y)) \\
| \\
\forall x\,\exists y\,\mathrm{ctrl}(x,y) \\
| \\
\neg\exists y\,\forall x\,\mathrm{ctrl}(x,y) \\
| \\
\exists y\,\mathrm{ctrl}(X,y) \\
| \\
\mathrm{ctrl}(X,c(X)) \\
| \\
\neg\forall x\,\mathrm{ctrl}(x,Y) \\
| \\
\neg\mathrm{ctrl}(d(Y),Y) \\
| \\
\text{not unifiable!}
\end{array}$$

Soundness of FOL Tableaux

Theorem (Soundness)

All provable (closed) FOL *formulas F are valid:*

$$\vdash_{tab\mathrm{FOL}} F \quad\Rightarrow\quad \vDash F$$

Soundness of FOL Tableaux

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All provable (closed) FOL *formulas F are valid:*

$$\vdash_{tab\mathrm{FOL}} F \quad \Rightarrow \quad \vDash F$$

Definition

An interpretation I is a *model* of tableau T iff for each assignment β there is a branch on which I, β satisfies all formulas.

Soundness of FOL Tableaux

Theorem (Soundness)

All provable (closed) FOL *formulas F are valid:*

$$\vdash_{tab\mathrm{FOL}} F \quad \Rightarrow \quad \vDash F$$

Proof.

□

Soundness of FOL Tableaux

Theorem (Soundness)

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Proof.

- Contrapositively, assume $\nvDash F$, then

$\square$

Soundness of FOL Tableaux

Theorem (Soundness)

All provable (closed) FOL *formulas F are valid:*

$$\vdash_{tab\mathrm{FOL}} F \quad \Rightarrow \quad \vDash F$$

Proof.

- Contrapositively, assume $\nvDash F$, then
- $\neg F$ has a model

☐

Soundness of FOL Tableaux

Theorem (Soundness)

All provable (closed) FOL *formulas F are valid:*

$$\vdash_{tab\mathrm{FOL}} F \quad \Rightarrow \quad \vDash F$$

Proof.

- Contrapositively, assume $\not\vDash F$, then
- $\neg F$ has a model
- Any tableau for F has a model: A tableau extending a tableau with a model, has a model too.

▸ Details

☐

Soundness of FOL Tableaux

Theorem (Soundness)

All provable (closed) FOL *formulas F are valid:*

$$\vdash_{tab\mathrm{FOL}} F \quad \Rightarrow \quad \vDash F$$

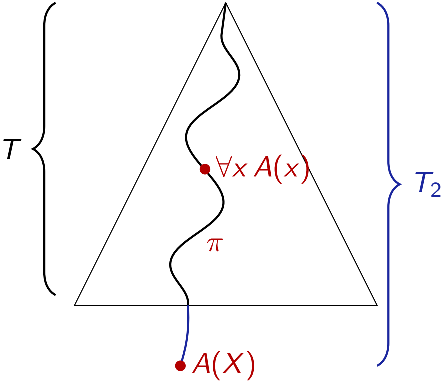
Proof.

- Contrapositively, assume $\not\vDash F$, then
- $\neg F$ has a model
- Any tableau for F has a model: A tableau extending a tableau with a model, has a model too. ▸ Details
- There is no closed tableau for F (closed tableaux have no models).

☐

Extensions of Tableaux with a Model have a Model

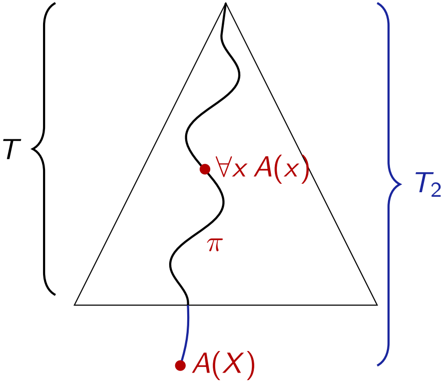
γ^*-extension



- I model of $T \Rightarrow I$ model of T_2:

Extensions of Tableaux with a Model have a Model

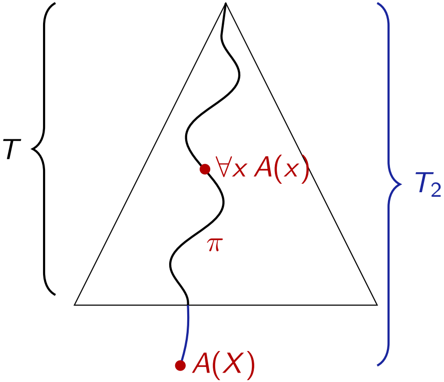
γ^*-extension



- I model of $T \Rightarrow I$ model of T_2:
- Let β be such that I, β satisfies all formulas on π (for other branches there is nothing to show).

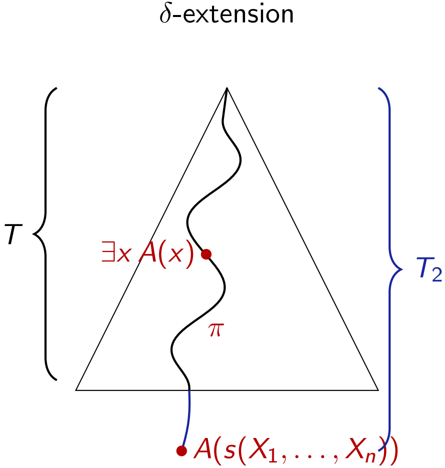
Extensions of Tableaux with a Model have a Model

γ^*-extension



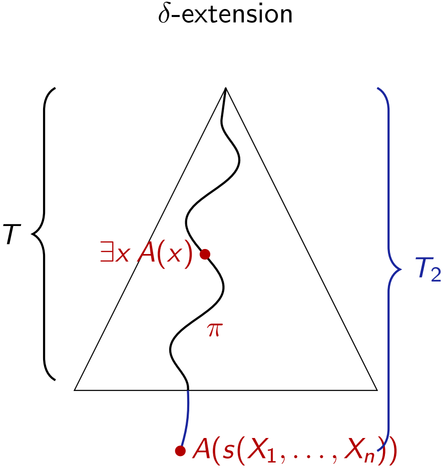
- I model of $T \Rightarrow I$ model of T_2:
- Let β be such that I, β satisfies all formulas on π (for other branches there is nothing to show).
- Now $I, \beta \models \forall x\, A(x)$ implies $I, \beta \models A(X)$.

Extensions of Tableaux with a Model have a Model



- I model of T $\Rightarrow$ there is a model I' of T_2 that only differs in the interpretation of new symbol s:

Extensions of Tableaux with a Model have a Model

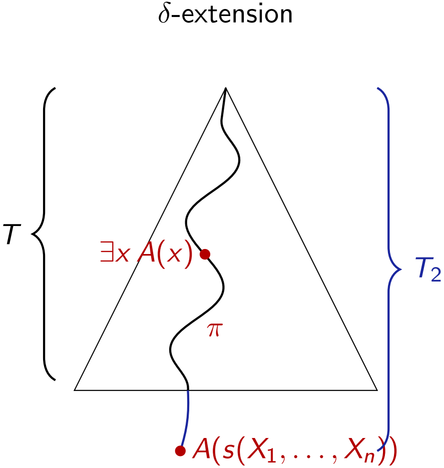


- I model of T $\Rightarrow$ there is a model I' of T_2 that only differs in the interpretation of new symbol s:
- $I'(s)(d_1,\dots,d_n) :=$

$$\begin{cases} d & \text{if } I,\beta \models \exists x\, A(x) \text{ for witness } d \\ e & \text{if } I,\beta \not\models \exists x\, A(x), \text{ arbitrary } e \end{cases}$$

for $\beta(X_1) = d_1, \dots, \beta(X_n) = d_n$, which are all free variables.

Extensions of Tableaux with a Model have a Model



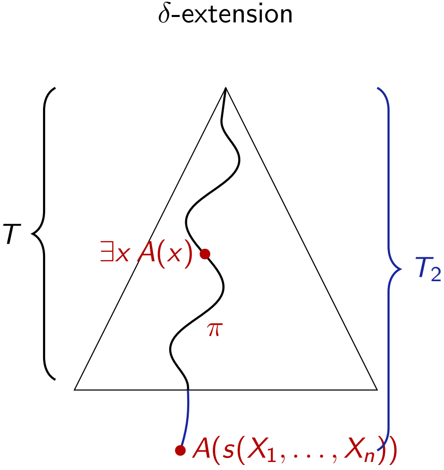
- I model of T $\Rightarrow$ there is a model I' of T_2 that only differs in the interpretation of new symbol s:
- $I'(s)(d_1, \ldots, d_n) :=$

$$\begin{cases} d & \text{if } I, \beta \models \exists x\, A(x) \text{ for witness } d \\ e & \text{if } I, \beta \not\models \exists x\, A(x), \text{ arbitrary } e \end{cases}$$

for $\beta(X_1) = d_1, \ldots, \beta(X_n) = d_n$, which are all free variables.

- Let β be such that I, β satisfies all formulas on π.

Extensions of Tableaux with a Model have a Model



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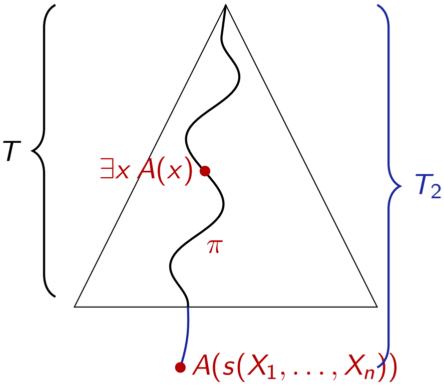
Extensions of Tableaux with a Model have a Model



δ-extension

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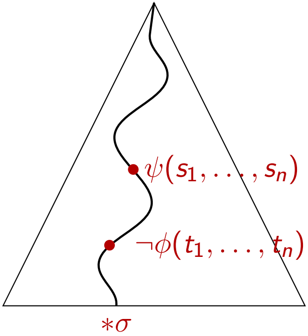
- s new, thus I', β satisfies π formulas

Closure of Tableaux with a Model have a Model

- I model of T $\Rightarrow$ I model of T_2:

path closure



Closure of Tableaux with a Model have a Model

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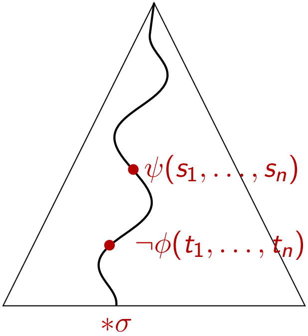

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Closure of Tableaux with a Model have a Model

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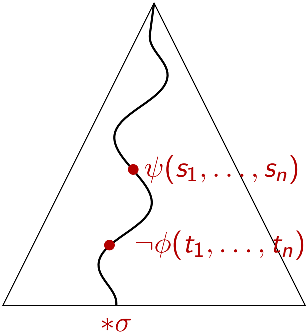


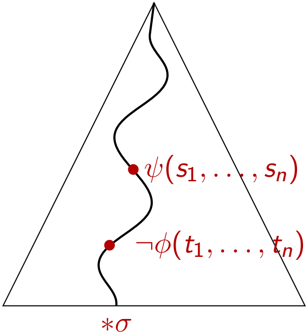
- I model of $T \;\Rightarrow\; I$ model of T_2:
- Let β be any assignment.
- By substitution lemma we have for all formulas G:

$$I, \sigma^*\beta \models G \text{ iff } I, \beta \models \sigma G$$

with $\sigma^*\beta(x) = [\![\sigma x]\!]_{I,\beta}$

Closure of Tableaux with a Model have a Model

path closure



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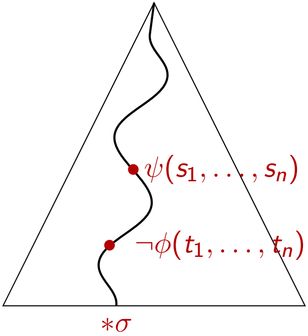

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Closure of Tableaux with a Model have a Model

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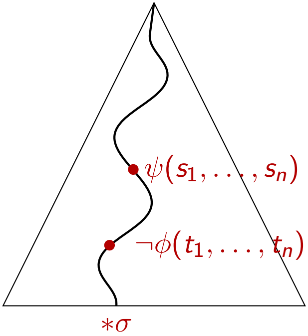

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  with $\sigma^*\beta(x) = [\![\sigma x]\!]_{I,\beta}$
- By premise, for $\sigma^*\beta$ there is a path π satisfied by $I, \sigma^*\beta$ in T.
- Thus I, β satisfies $\sigma\pi$ in T_2.
- $\sigma\pi$ belongs to T_2, because
  $I, \beta \models \sigma(\psi(s_1, \dots, s_n))$ contradicts
  $I, \beta \models \sigma(\neg\phi(t_1, \dots, t_n))$

Repetitive γ^* Tableau Expansions

$$\big(\mathrm{safe}(0) \land \forall x\,(\mathrm{safe}(x) \to \mathrm{safe}(n(x))) \to \mathrm{safe}(n(n(0)))\big)$$

▸ Rules

Repetitive γ^* Tableau Expansions

$$\neg\big(\mathrm{safe}(0) \land \forall x\, (\mathrm{safe}(x) \rightarrow \mathrm{safe}(n(x))) \rightarrow \mathrm{safe}(n(n(0)))\big)$$

Rules

Repetitive γ^* Tableau Expansions

$$\neg\big(\mathrm{safe}(0) \land \forall x\,(\mathrm{safe}(x) \to \mathrm{safe}(n(x))) \to \mathrm{safe}(n(n(0)))\big)$$

$$\mathrm{safe}(0)$$

▸ Rules

Repetitive γ^* Tableau Expansions

$$\neg\big(\mathrm{safe}(0) \land \forall x\,(\mathrm{safe}(x) \rightarrow \mathrm{safe}(n(x))) \rightarrow \mathrm{safe}(n(n(0)))\big)$$

$$|$$

$$\mathrm{safe}(0)$$

$$|$$

$$\forall x\,(\mathrm{safe}(x) \rightarrow \mathrm{safe}(n(x)))$$

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$$|$$

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$$|$$

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$$|$$

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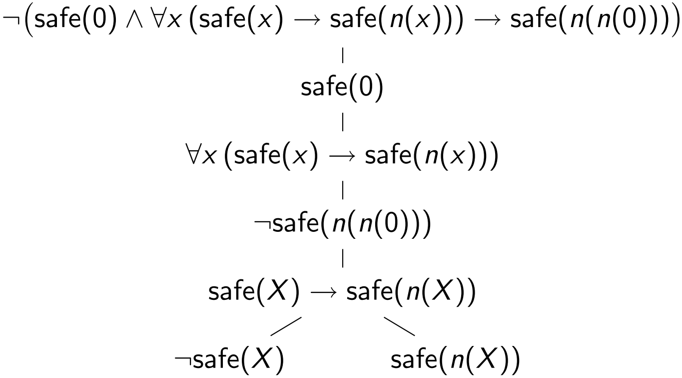
▸ Rules

Repetitive γ^* Tableau Expansions



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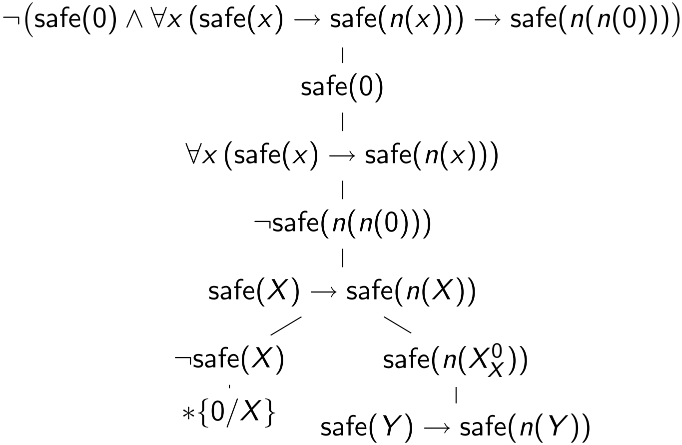
$$\neg\mathrm{safe}(n(n(0)))$$

$$\mathrm{safe}(X) \to \mathrm{safe}(n(X))$$

$$\neg\mathrm{safe}(X) \qquad\qquad \mathrm{safe}(n(X))$$

Rules

Repetitive γ^* Tableau Expansions



$\neg\big(\mathsf{safe}(0) \land \forall x\,(\mathsf{safe}(x) \to \mathsf{safe}(n(x))) \to \mathsf{safe}(n(n(0)))\big)$

$\mathsf{safe}(0)$

$\forall x\,(\mathsf{safe}(x) \to \mathsf{safe}(n(x)))$

$\neg\mathsf{safe}(n(n(0)))$

$\mathsf{safe}(X) \to \mathsf{safe}(n(X))$

$\neg\mathsf{safe}(X)$

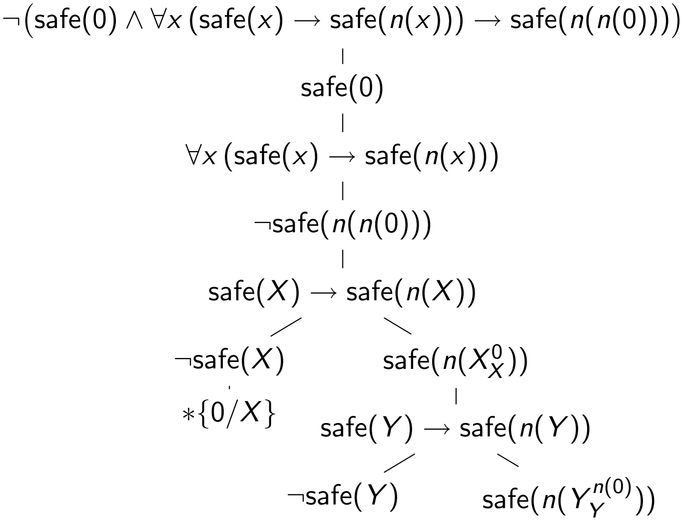
$*\{0/X\}$

$\mathsf{safe}(n(X_X^0))$

$\mathsf{safe}(Y) \to \mathsf{safe}(n(Y))$

Rules

Repetitive γ^* Tableau Expansions



$$\neg\big(\text{safe}(0) \land \forall x\,(\text{safe}(x) \to \text{safe}(n(x))) \to \text{safe}(n(n(0)))\big)$$

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$$\neg\text{safe}(n(n(0)))$$

$$\text{safe}(X) \to \text{safe}(n(X))$$

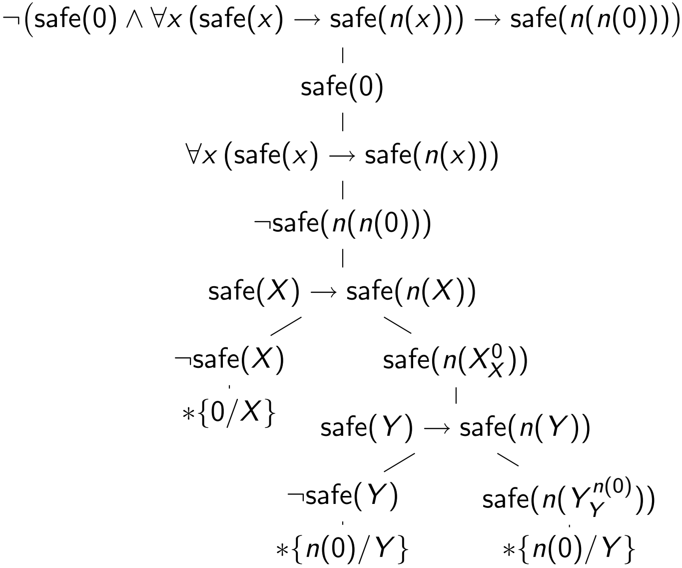
$$\neg\text{safe}(X) \qquad \text{safe}(n(X_X^0))$$

$$*\{0/X\} \qquad \text{safe}(Y) \to \text{safe}(n(Y))$$

$$\neg\text{safe}(Y) \qquad \text{safe}(n(Y_Y^{n(0)}))$$

Rules

Repetitive γ^* Tableau Expansions



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$$\text{safe}(0)$$

$$\forall x\,(\text{safe}(x) \to \text{safe}(n(x)))$$

$$\neg\text{safe}(n(n(0)))$$

$$\text{safe}(X) \to \text{safe}(n(X))$$

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$$*\{0/X\}$$

$$\text{safe}(n(X_X^0))$$

$$\text{safe}(Y) \to \text{safe}(n(Y))$$

$$\neg\text{safe}(Y)$$

$$*\{n(0)/Y\}$$

$$\text{safe}(n(Y_Y^{n(0)}))$$

$$*\{n(0)/Y\}$$

▸ Rules

Repetitive γ^* Tableau Expansions

Can we limit the number of γ applications needed in proofs?

Repetitive γ^* Tableau Expansions

Can we limit the number of γ applications needed in proofs?

NO!

Repetitive γ^* Tableau Expansions

Can we limit the number of γ applications needed in proofs?

NO!

Can we prove everything at all?

Completeness of FOL Tableaux

Theorem (Completeness)

All valid FOL *formulas F are provable:*

$$\vDash F \quad \Rightarrow \quad \vdash_{tab\mathrm{FOL}} F$$

M. D'Agostino, D. Gabbay, R. Hähnle, and J. Posegga, editors.
Handbook of Tableau Methods (in 3 volumes).
Kluwer, 1999.

M. Fitting.
First-Order Logic and Automated Theorem Proving.
Springer, New York, 2nd edition, 1996.

J. A. Robinson and A. Voronkov, editors.
Handbook of Automated Reasoning (in 2 volumes).
Elsevier and MIT Press, 2001.

R. M. Smullyan.
First-Order Logic.
Dover, 1995.