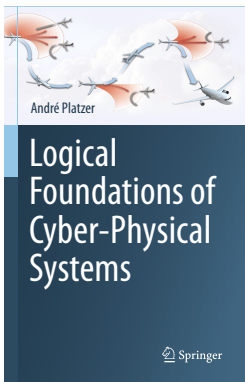
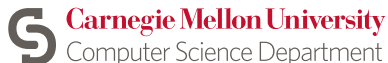


16: Winning & Proving Hybrid Games

Logical Foundations of Cyber-Physical Systems



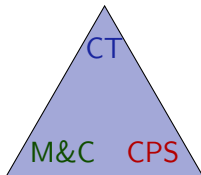
André Platzer



- 1 Learning Objectives
- 2 Semantical Considerations
- 3 Dynamic Axioms for Hybrid Games
 - Assignments
 - Differential Equations
 - Challenge Games
 - Choice Games
 - Sequential Games
 - Dual Games
 - Example Proof: Demon's Choice
- 4 Repetitions
 - Proofs for Loops
 - Example Proof: Dual Filibuster
 - Example Proof: Push-around Cart
- 5 Axiomatization
- 6 Summary

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rigorous reasoning for adversarial dynamics
compositional reasoning from compositional semantics
modular addition of adversarial dynamics
axiomatization of dGL



analytical&semantical interaction
discrete+continuous+adversarial
fixpoints

CPS semantics
align semantics&reasoning
operational CPS effects

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Definition (Hybrid game α)

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x P \mid \exists x P \mid \langle \alpha \rangle P \mid [\alpha] P$$

Discrete
Assign

Test
Game

Differential
Equation

Choice
Game

Seq.
Game

Repeat
Game

Definition (Hybrid game α)

$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$

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$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x P \mid \exists x P \mid \langle \alpha \rangle P \mid [\alpha] P$

All
Reals

Some
Reals

Discrete
Assign

Test
Game

Differential
Equation

Choice
Game

Seq.
Game

Repeat
Game

Dual
Game

Definition (Hybrid game α)

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All
Reals

Some
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Assign

Test
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All
Reals

Some
Reals

Angel
Wins

Discrete
Assign

Test
Game

Differential
Equation

Choice
Game

Seq.
Game

Repeat
Game

Dual
Game

Definition (Hybrid game α)

$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$

Definition (dGL Formula P)

$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x P \mid \exists x P \mid \langle \alpha \rangle P \mid [\alpha] P$

All
Reals

Some
Reals

Angel
Wins

Demon
Wins

Discrete Assign

Test Game

Differential Equation

Choice Game

Seq. Game

Repeat Game

Dual Game

Definition (Hybrid game α)

$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$

Definition (dGL Formula P)

$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x P \mid \exists x P \mid \langle \alpha \rangle P \mid [\alpha] P$

“Angel has Wings $\langle \alpha \rangle$ ”

All Reals

Some Reals

Angel Wins

Demon Wins

Definition (Hybrid game α) $[[\cdot]] : \text{HG} \rightarrow (\wp(\mathcal{S}) \rightarrow \wp(\mathcal{S}))$

$$\begin{aligned}
s_{x := e}(X) &= \{\omega \in \mathcal{S} : \omega_x^{\omega[e]} \in X\} \\
s_{x' = f(x)}(X) &= \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X \text{ for some } \varphi: [0, r] \rightarrow \mathcal{S}, \varphi \models x' = f(x)\} \\
s_{?Q}(X) &= [[Q]] \cap X \\
s_{\alpha \cup \beta}(X) &= s_{\alpha}(X) \cup s_{\beta}(X) \\
s_{\alpha; \beta}(X) &= s_{\alpha}(s_{\beta}(X)) \\
s_{\alpha^*}(X) &= \bigcap \{Z \subseteq \mathcal{S} : X \cup s_{\alpha}(Z) \subseteq Z\} \\
s_{\alpha^d}(X) &= (s_{\alpha}(X^c))^c
\end{aligned}$$

Definition (dGL Formula P) $[[\cdot]] : \text{Fml} \rightarrow \wp(\mathcal{S})$

$$\begin{aligned}
[[e_1 \geq e_2]] &= \{\omega \in \mathcal{S} : \omega[e_1] \geq \omega[e_2]\} \\
[[\neg P]] &= ([[P]])^c \\
[[P \wedge Q]] &= [[P]] \cap [[Q]] \\
[[\langle \alpha \rangle P]] &= s_{\alpha}([[P]]) \\
[[[\alpha] P]] &= \delta_{\alpha}([[P]])
\end{aligned}$$

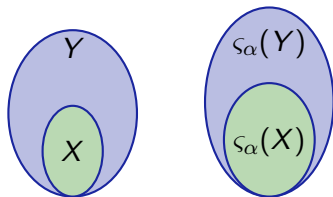


Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\models \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Lemma (Monotonicity)

$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ and $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ for all $X \subseteq Y$



Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\models \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

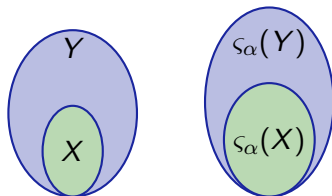
Corollary

Determined: At least one player wins: $\neg\langle\alpha\rangle\neg P \rightarrow [\alpha]P$ so $\langle\alpha\rangle\neg P \vee [\alpha]P$

Consistent: At most one player wins: $[\alpha]P \rightarrow \neg\langle\alpha\rangle\neg P$ so $\neg([\alpha]P \wedge \langle\alpha\rangle\neg P)$

Lemma (Monotonicity)

$s_\alpha(X) \subseteq s_\alpha(Y)$ and $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ for all $X \subseteq Y$



Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\models \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Proof Sketch.

$$\varsigma_{\alpha\cup\beta}(X^{\mathbb{C}})^{\mathbb{C}} = (\varsigma_{\alpha}(X^{\mathbb{C}}) \cup \varsigma_{\beta}(X^{\mathbb{C}}))^{\mathbb{C}} = \varsigma_{\alpha}(X^{\mathbb{C}})^{\mathbb{C}} \cap \varsigma_{\beta}(X^{\mathbb{C}})^{\mathbb{C}} = \delta_{\alpha}(X) \cap \delta_{\beta}(X) = \delta_{\alpha\cup\beta}(X) \quad \square$$

Lemma (Monotonicity)

$\varsigma_{\alpha}(X) \subseteq \varsigma_{\alpha}(Y)$ and $\delta_{\alpha}(X) \subseteq \delta_{\alpha}(Y)$ for all $X \subseteq Y$

Proof Sketch.

- $X \subseteq Y$ so $X^{\mathbb{C}} \supseteq Y^{\mathbb{C}}$ so $\varsigma_{\alpha}(X^{\mathbb{C}}) \supseteq \varsigma_{\alpha}(Y^{\mathbb{C}})$ so $\varsigma_{\alpha^d}(X) = (\varsigma_{\alpha}(X^{\mathbb{C}}))^{\mathbb{C}} \subseteq (\varsigma_{\alpha}(Y^{\mathbb{C}}))^{\mathbb{C}} = \varsigma_{\alpha^d}(Y)$.
- $\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} : X \cup \varsigma_{\alpha}(Z) \subseteq Z\} \subseteq \bigcap\{Z \subseteq \mathcal{S} : Y \cup \varsigma_{\alpha}(Z) \subseteq Z\} = \varsigma_{\alpha^*}(Y)$ because $X \subseteq Y$ □

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\models \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Lemma (Monotonicity)

$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ and $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ for all $X \subseteq Y$

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\models \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Corollary (Axiom: Determinacy)

$[\cdot] \quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

Lemma (Monotonicity)

$s_\alpha(X) \subseteq s_\alpha(Y)$ and $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ for all $X \subseteq Y$

Corollary (Rule: Monotonicity)

$$M \quad \frac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q} \qquad M[\cdot] \quad \frac{P \rightarrow Q}{[\alpha]P \rightarrow [\alpha]Q}$$



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Axiom (Assignment)

TOCL'15

$\langle := \rangle \quad \langle x := e \rangle p(x) \leftrightarrow$

$\langle x := e \rangle (X)$



$$\langle := \rangle \quad \langle x := e \rangle p(x) \leftrightarrow p(e)$$

$s_{x:=e}(X)$

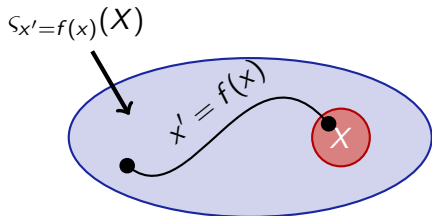


Axiom (Differential Equation)

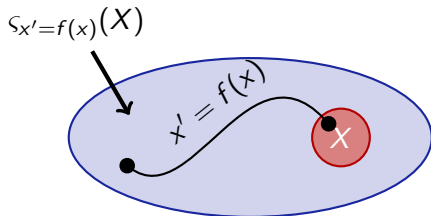
TOCL'15

$\langle \langle ' \rangle \langle x' = f(x) \rangle \rangle p(x) \leftrightarrow$

$(y'(t) = f(y))$



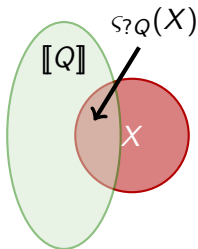
$$\langle \langle \rangle \langle x' = f(x) \rangle p(x) \leftrightarrow \exists t \geq 0 \langle x := y(t) \rangle p(x) \quad (y'(t) = f(y))$$



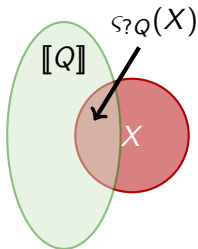
Axiom (Test / Challenge)

TOCL'15

$\langle ? \rangle \langle ?Q \rangle P \leftrightarrow$



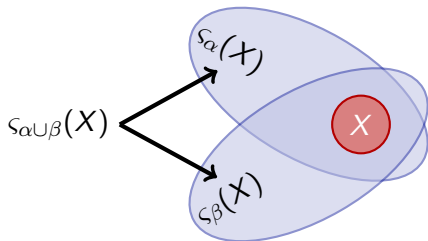
$$\langle ? \rangle \langle ?Q \rangle P \leftrightarrow Q \wedge P$$



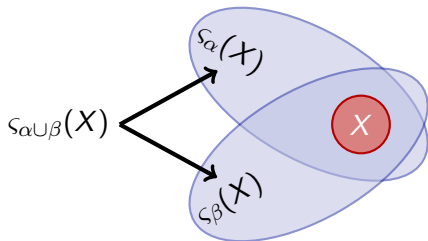
Axiom (Choice Game)

TOCL'15

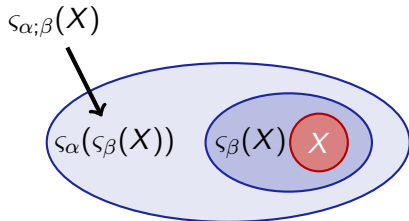
$\langle U \rangle \langle \alpha \cup \beta \rangle P \leftrightarrow$



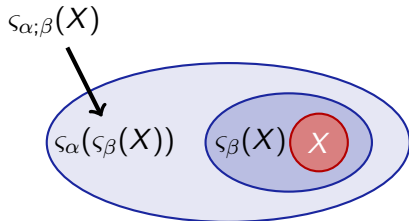
$$\langle U \rangle \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$



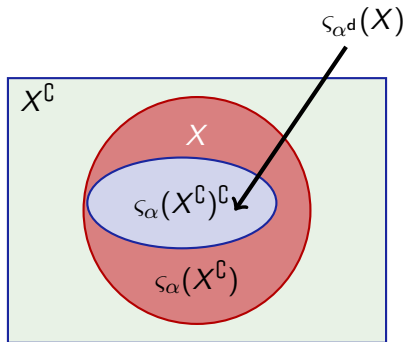
$$\langle ; \rangle \langle \alpha; \beta \rangle P \leftrightarrow$$



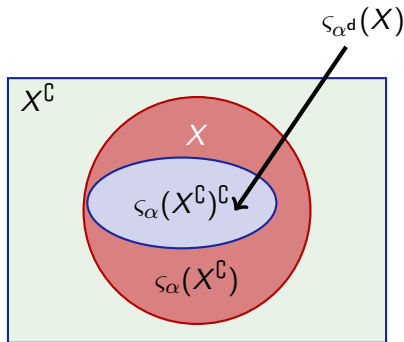
$$\langle ; \rangle \langle \alpha; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P$$



$$\langle^d \rangle \quad \langle \alpha^d \rangle P \leftrightarrow$$



$$\langle^d \rangle \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$



$$\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow$$

$$\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P$$

$$\alpha \cap \beta \equiv (\alpha^d \cup \beta^d)^d$$

$$\frac{\langle^d \rangle \frac{}{\vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}}{\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}}$$

$$\langle^d \rangle \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\frac{\langle^U \rangle \frac{}{\vdash \neg \langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}}{\vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}}{\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}$$

$$\langle U \rangle \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$

$$\begin{array}{c} \langle^d \rangle \frac{}{\vdash \neg(\langle \alpha^d \rangle \neg P \vee \langle \beta^d \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P} \\ \langle U \rangle \frac{}{\vdash \neg \langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P} \\ \langle^d \rangle \frac{}{\vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P} \\ \hline \vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \end{array}$$

$$\langle^d \rangle \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\frac{}{\vdash \neg(\neg \langle \alpha \rangle \neg \neg P \vee \neg \langle \beta \rangle \neg \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}$$

$$\frac{\langle^d \rangle \vdash \neg(\langle \alpha^d \rangle \neg P \vee \langle \beta^d \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}{\langle \cup \rangle \vdash \neg \langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}$$

$$\frac{\langle^d \rangle \vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}{\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P}$$

$$\begin{array}{c}
 \hline
 \vdash \langle \alpha \rangle P \wedge \langle \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \neg(\neg\langle \alpha \rangle \neg\neg P \vee \neg\langle \beta \rangle \neg\neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \neg(\langle \alpha^d \rangle \neg P \vee \langle \beta^d \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle \cup \rangle \vdash \neg\langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P
 \end{array}$$

$$\begin{array}{c}
 * \\
 \hline
 \vdash \langle \alpha \rangle P \wedge \langle \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \neg(\neg\langle \alpha \rangle \neg\neg P \vee \neg\langle \beta \rangle \neg\neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \neg(\langle \alpha^d \rangle \neg P \vee \langle \beta^d \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle \cup \rangle \vdash \neg\langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P
 \end{array}$$

$$\begin{array}{c}
 * \\
 \hline
 \vdash \langle \alpha \rangle P \wedge \langle \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \neg(\neg\langle \alpha \rangle \neg\neg P \vee \neg\langle \beta \rangle \neg\neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \neg(\langle \alpha^d \rangle \neg P \vee \langle \beta^d \rangle \neg P) \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle \cup \rangle \vdash \neg\langle \alpha^d \cup \beta^d \rangle \neg P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \langle^d \rangle \vdash \langle (\alpha^d \cup \beta^d)^d \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P \\
 \hline
 \vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P
 \end{array}$$

Derived axiom:

$$\langle \cap \rangle \quad \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P$$

$$\boxed{\cdot} \overline{\vdash [\alpha \cap \beta]P \leftrightarrow}$$

$$\boxed{\cdot} \frac{}{\vdash [\alpha \wedge \beta]P \leftrightarrow [\alpha]P \vee [\beta]P}$$

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\frac{\langle\cap\rangle \frac{}{\vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P}}{[\cdot] \frac{}{\vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P}}$$

$$\langle \cap \rangle \quad \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P$$

$$\frac{\frac{\frac{\overline{\vdash \neg(\langle \alpha \rangle \neg P \wedge \langle \beta \rangle \neg P) \leftrightarrow [\alpha] P \vee [\beta] P}}{\langle \cap \rangle \vdash \neg \langle \alpha \cap \beta \rangle \neg P \leftrightarrow [\alpha] P \vee [\beta] P}}{[\cdot] \vdash [\alpha \cap \beta] P \leftrightarrow [\alpha] P \vee [\beta] P}}$$

$$\begin{array}{c}
 \frac{[\cdot] \quad \overline{\vdash \neg\langle\alpha\rangle\neg P \vee \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P}}{\vdash \neg(\langle\alpha\rangle\neg P \wedge \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \vee [\beta]P} \\
 \langle\cap\rangle \quad \overline{\vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P} \\
 [\cdot] \quad \overline{\vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P}
 \end{array}$$

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\frac{}{\vdash [\alpha]P \vee [\beta]P \leftrightarrow [\alpha]P \vee [\beta]P}$$

$$[\cdot] \frac{}{\vdash \neg\langle\alpha\rangle\neg P \vee \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P}$$

$$\frac{}{\vdash \neg(\langle\alpha\rangle\neg P \wedge \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \vee [\beta]P}$$

$$\langle\cap\rangle \frac{}{\vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P}$$

$$[\cdot] \frac{}{\vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P}$$

$$\begin{array}{c}
 * \\
 \hline
 \vdash [\alpha]P \vee [\beta]P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 [\cdot] \vdash \neg \langle \alpha \rangle \neg P \vee \neg \langle \beta \rangle \neg P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 \vdash \neg (\langle \alpha \rangle \neg P \wedge \langle \beta \rangle \neg P) \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 \langle \cap \rangle \vdash \neg \langle \alpha \cap \beta \rangle \neg P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 [\cdot] \vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P
 \end{array}$$

$$\begin{array}{c}
 * \\
 \hline
 \vdash [\alpha]P \vee [\beta]P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 [\cdot] \vdash \neg \langle \alpha \rangle \neg P \vee \neg \langle \beta \rangle \neg P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 \vdash \neg (\langle \alpha \rangle \neg P \wedge \langle \beta \rangle \neg P) \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 \langle \cap \rangle \vdash \neg \langle \alpha \cap \beta \rangle \neg P \leftrightarrow [\alpha]P \vee [\beta]P \\
 \hline
 [\cdot] \vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P
 \end{array}$$

Derived axioms:

$$[\cap] [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P$$

$$\langle \cap \rangle \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \wedge \langle \beta \rangle P$$

$$[\cdot] [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\langle := \rangle \langle x := e \rangle p(x) \leftrightarrow p(e)$$

$$\langle ' \rangle \langle x' = f(x) \rangle p(x) \leftrightarrow \exists t \geq 0 \langle x := y(t) \rangle p(x)$$

$$\langle ? \rangle \langle ?Q \rangle P \leftrightarrow Q \wedge P$$

$$\langle \cup \rangle \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$

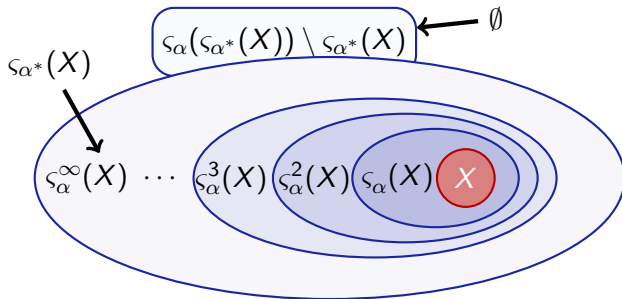
$$\langle ; \rangle \langle \alpha ; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P$$

$$\langle ^d \rangle \langle \alpha^d \rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

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Definition (Hybrid game α)

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Proof

$$\begin{array}{l} \vdash P \rightarrow [\alpha]P \\ \hline \vdash P \rightarrow P \wedge [\alpha]P \\ [\cdot] \vdash P \rightarrow P \wedge \neg \langle \alpha \rangle \neg P \\ \hline \vdash \neg P \vee \langle \alpha \rangle \neg P \rightarrow \neg P \\ FP \vdash \langle \alpha^* \rangle \neg P \rightarrow \neg P \\ \hline \vdash P \rightarrow \neg \langle \alpha^* \rangle \neg P \\ [\cdot] \vdash P \rightarrow [\alpha^*]P \end{array}$$

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\langle := \rangle \quad \langle x := e \rangle p(x) \leftrightarrow p(e)$$

$$\langle ' \rangle \quad \langle x' = f(x) \rangle p(x) \leftrightarrow \exists t \geq 0 \langle x := y(t) \rangle p(x)$$

$$\langle ? \rangle \quad \langle ?Q \rangle P \leftrightarrow Q \wedge P$$

$$\langle \cup \rangle \quad \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$

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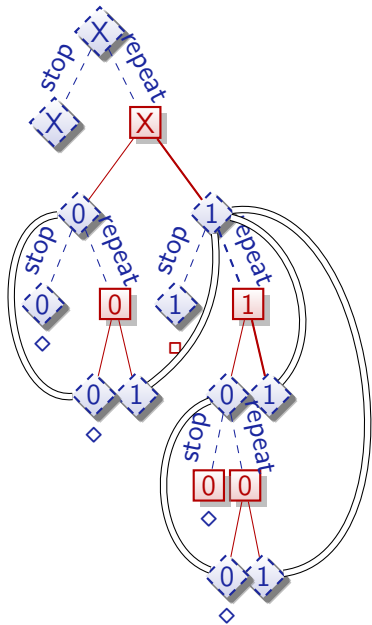
$$\langle * \rangle \quad \langle \alpha^* \rangle P \leftrightarrow P \vee \langle \alpha \rangle \langle \alpha^* \rangle P$$

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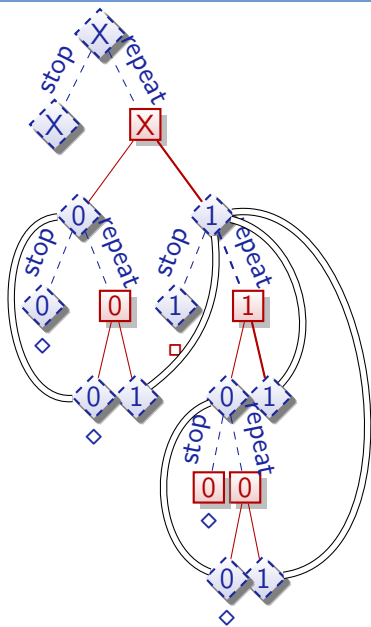
$$\text{M} \quad \frac{P \rightarrow Q}{\langle \alpha \rangle P \rightarrow \langle \alpha \rangle Q}$$

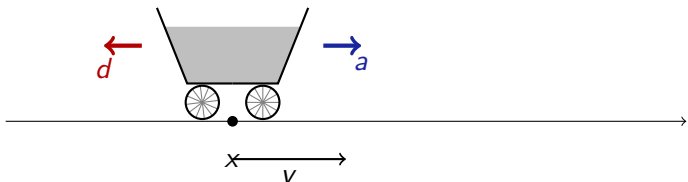
$$\text{FP} \quad \frac{P \vee \langle \alpha \rangle Q \rightarrow Q}{\langle \alpha^* \rangle P \rightarrow Q}$$

$$\langle^d \rangle \frac{}{x = 0 \vdash \langle (x := 0 \cup x := 1)^x \rangle x = 0}$$



\mathbb{R}	$x = 0 \vdash 0 = 0 \vee 1 = 0$
$\langle := \rangle$	$x = 0 \vdash \langle x := 0 \rangle x = 0 \vee \langle x := 1 \rangle x = 0$
$\langle \cup \rangle$	$x = 0 \vdash \langle x := 0 \cup x := 1 \rangle x = 0$
$\langle ^d \rangle$	$x = 0 \vdash \neg \langle x := 0 \cap x := 1 \rangle \neg x = 0$
$\llbracket \cdot \rrbracket$	$x = 0 \vdash \llbracket x := 0 \cap x := 1 \rrbracket x = 0$
ind	$x = 0 \vdash \llbracket (x := 0 \cap x := 1)^* \rrbracket x = 0$
$\langle ^d \rangle$	$x = 0 \vdash \langle (x := 0 \cup x := 1)^x \rangle x = 0$





$$\text{ind } \overline{J \vdash [((d := 1 \wedge d := -1); (a := 1 \vee a := -1); \{x' = v, v' = a + d\})^*] x \geq 0}$$

$$\begin{array}{l} \text{[i]} \\ \text{ind} \end{array} \frac{J \vdash [(d := 1 \wedge d := -1); (a := 1 \vee a := -1); \{x' = v, v' = a + d\}]J}{J \vdash [((d := 1 \wedge d := -1); (a := 1 \vee a := -1); \{x' = v, v' = a + d\})^*]x \geq 0}$$

$$\begin{array}{l}
 [\wedge] \frac{J \vdash [d := 1 \wedge d := -1] [(a := 1 \vee a := -1); \{x' = v, v' = a + d\}] J}{J \vdash [(d := 1 \wedge d := -1); (a := 1 \vee a := -1); \{x' = v, v' = a + d\}] J} \\
 [!] \frac{J \vdash [(d := 1 \wedge d := -1); (a := 1 \vee a := -1); \{x' = v, v' = a + d\}] J}{J \vdash [((d := 1 \wedge d := -1); (a := 1 \vee a := -1); \{x' = v, v' = a + d\})^*] x \geq 0} \\
 \text{ind}
 \end{array}$$

$$\begin{array}{l}
 \text{VR,WR} \\
 \text{[}\cap\text{]} \\
 \text{[!]} \\
 \text{ind}
 \end{array}
 \frac{
 \frac{
 \frac{
 J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}] J \vee [d := -1] \dots
 }{
 J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}] J
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 }$$

$$\begin{array}{l}
 \text{[:=]} \frac{J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J}{\vdots} \\
 \text{VR, WR} \frac{J \vdash [d := 1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J \vee [d := -1] \dots}{\vdots} \\
 \text{[}\cap\text{]} \frac{J \vdash [d := 1 \cap d := -1][(a := 1 \cup a := -1); \{x' = v, v' = a + d\}]J}{\vdots} \\
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 \end{array}$$

$$\begin{array}{l}
 [i] \frac{}{J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J} \\
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 \end{array}$$

[U]	$J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a + 1\}]J$
[i]	$J \vdash [(a := 1 \cup a := -1); \{x' = v, v' = a + 1\}]J$
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 J \vdash [\{x' = v, v' = 1 + 1\}]J \wedge [\{x' = v, v' = -1 + 1\}]J \\
 \hline
 [:=] \quad J \vdash [a := 1][\{x' = v, v' = a + 1\}]J \wedge [a := -1][\{x' = v, v' = a + 1\}]J \\
 \hline
 [\cup] \quad J \vdash [a := 1 \cup a := -1][\{x' = v, v' = a + 1\}]J \\
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 \hline
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 \hline
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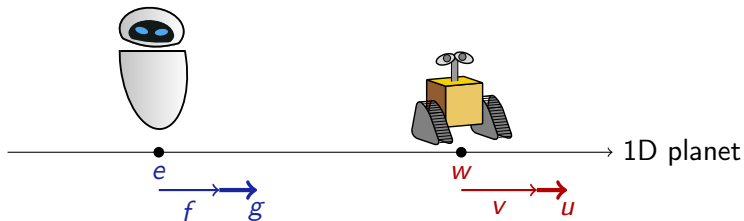
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 \end{array}$$

$$J \stackrel{\text{def}}{\equiv} x \geq 0 \wedge v \geq 0$$

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\text{ind}
\end{array}$$

$$J \stackrel{\text{def}}{=} x \geq 0 \wedge v \geq 0 \quad ['], [:=] \frac{x \geq 0 \wedge v \geq 0 \vdash \forall t \geq 0 (x + vt + t^2 \geq 0 \wedge v + 2t \geq 0)}{J \vdash [\{x' = v, v' = 1 + 1\}]J}$$

$$['], [:=] \frac{x \geq 0 \wedge v \geq 0 \vdash \forall t \geq 0 (x + vt \geq 0 \wedge v \geq 0)}{J \vdash [\{x' = v, v' = 0\}]J}$$



$$(w - e)^2 \leq 1 \wedge v = f \rightarrow$$

$$\langle ((u := 1 \cap u := -1);$$

$$(g := 1 \cup g := -1);$$

$$t := 0;$$

$$\{w' = v, v' = u, e' = f, f' = g, t' = 1 \& t \leq 1\}^d$$

$$\rangle^x \rangle (w - e)^2 \leq 1$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u



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$$[\cdot] \quad [\alpha]P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\langle := \rangle \quad \langle x := e \rangle p(x) \leftrightarrow p(e)$$

$$\langle ' \rangle \quad \langle x' = f(x) \rangle p(x) \leftrightarrow \exists t \geq 0 \langle x := y(t) \rangle p(x)$$

$$\langle ? \rangle \quad \langle ?Q \rangle P \leftrightarrow Q \wedge P$$

$$\langle \cup \rangle \quad \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$

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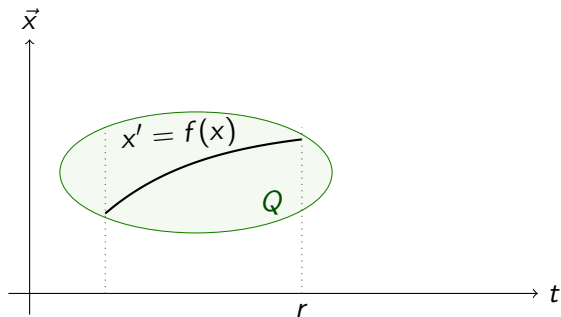
$$\langle ^d \rangle \quad \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\text{M} \quad \frac{P \rightarrow Q}{\langle \alpha \rangle P \rightarrow \langle \alpha \rangle Q}$$

$$\text{FP} \quad \frac{P \vee \langle \alpha \rangle Q \rightarrow Q}{\langle \alpha^* \rangle P \rightarrow Q}$$

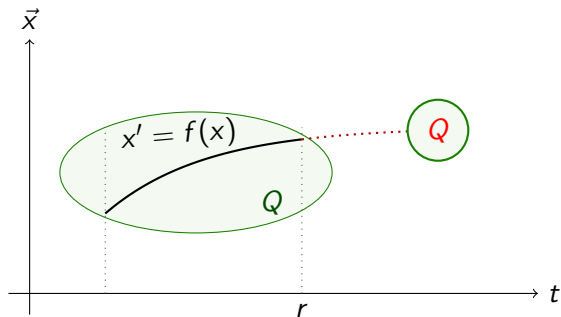
$$x' = f(x) \ \& \ Q$$

$$x' = f(x); \ ?(Q)$$



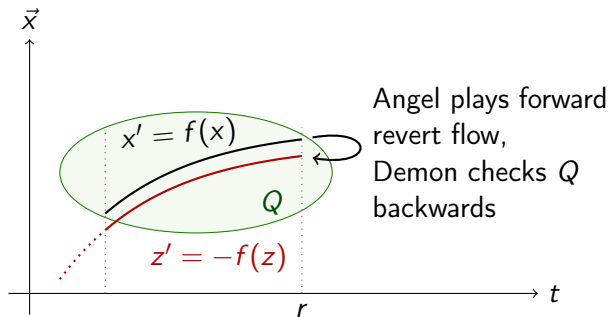
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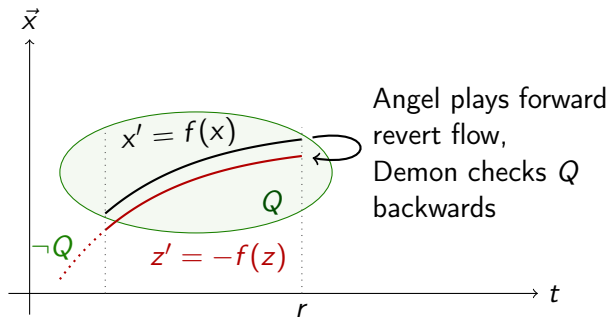
$$x' = f(x) \ \& \ Q$$

$$x' = f(x); (z := x; z' = -f(z))^d; ?(Q(z))$$

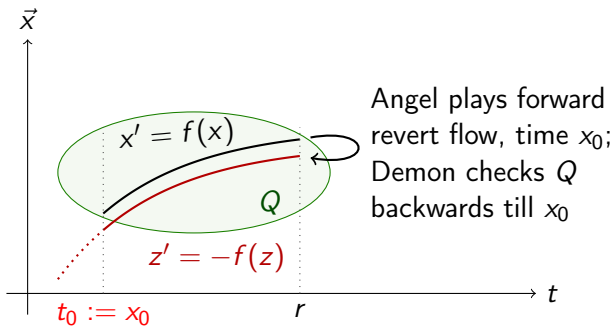


$$x' = f(x) \ \& \ Q$$

$$x' = f(x); (z := x; z' = -f(z))^d; ?(Q(z))$$

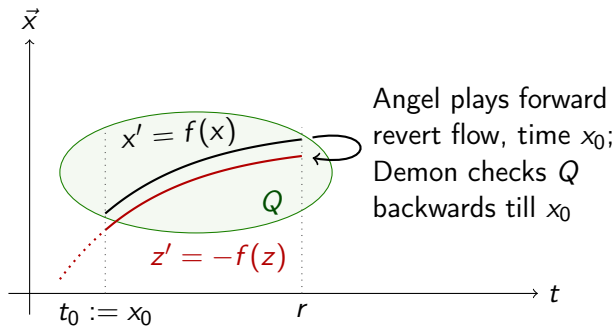


$$x' = f(x) \ \& \ Q \equiv t_0 := x_0; x' = f(x); (z := x; z' = -f(z))^d; ?(z_0 \geq t_0 \rightarrow Q(z))$$



A “There and Back Again” Game

$x' = f(x)$ & $Q \equiv t_0 := x_0; x' = f(x); (z := x; z' = -f(z))^d; ?(z_0 \geq t_0 \rightarrow Q(z))$



Lemma

Evolution domains definable by games

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Definition (Hybrid game α) $[[\cdot]] : \text{HG} \rightarrow (\wp(\mathcal{S}) \rightarrow \wp(\mathcal{S}))$

$$\begin{aligned}
s_{x:=e}(X) &= \{\omega \in \mathcal{S} : \omega_x^{\omega[e]} \in X\} \\
s_{x'=f(x)}(X) &= \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X \text{ for some } \varphi:[0,r] \rightarrow \mathcal{S}, \varphi \models x' = f(x)\} \\
s_{?Q}(X) &= [[Q]] \cap X \\
s_{\alpha \cup \beta}(X) &= s_{\alpha}(X) \cup s_{\beta}(X) \\
s_{\alpha;\beta}(X) &= s_{\alpha}(s_{\beta}(X)) \\
s_{\alpha^*}(X) &= \bigcap \{Z \subseteq \mathcal{S} : X \cup s_{\alpha}(Z) \subseteq Z\} \\
s_{\alpha^d}(X) &= (s_{\alpha}(X^c))^c
\end{aligned}$$

Definition (dGL Formula P) $[[\cdot]] : \text{Fml} \rightarrow \wp(\mathcal{S})$

$$\begin{aligned}
[[e_1 \geq e_2]] &= \{\omega \in \mathcal{S} : \omega[e_1] \geq \omega[e_2]\} \\
[[\neg P]] &= ([[P]])^c \\
[[P \wedge Q]] &= [[P]] \cap [[Q]] \\
[[\langle \alpha \rangle P]] &= s_{\alpha}([[P]]) \\
[[[\alpha] P]] &= \delta_{\alpha}([[P]])
\end{aligned}$$

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

$$\langle := \rangle \quad \langle x := e \rangle p(x) \leftrightarrow p(e)$$

$$\langle ' \rangle \quad \langle x' = f(x) \rangle p(x) \leftrightarrow \exists t \geq 0 \langle x := y(t) \rangle p(x)$$

$$\langle ? \rangle \quad \langle ?Q \rangle P \leftrightarrow Q \wedge P$$

$$\langle \cup \rangle \quad \langle \alpha \cup \beta \rangle P \leftrightarrow \langle \alpha \rangle P \vee \langle \beta \rangle P$$

$$\langle ; \rangle \quad \langle \alpha ; \beta \rangle P \leftrightarrow \langle \alpha \rangle \langle \beta \rangle P$$

$$\langle * \rangle \quad \langle \alpha^* \rangle P \leftrightarrow P \vee \langle \alpha \rangle \langle \alpha^* \rangle P$$

$$\langle ^d \rangle \quad \langle \alpha^d \rangle P \leftrightarrow \neg \langle \alpha \rangle \neg P$$

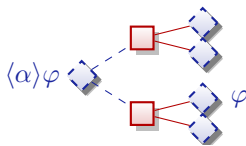
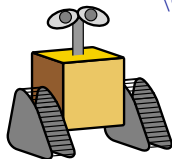
$$\text{M} \quad \frac{P \rightarrow Q}{\langle \alpha \rangle P \rightarrow \langle \alpha \rangle Q}$$

$$\text{FP} \quad \frac{P \vee \langle \alpha \rangle Q \rightarrow Q}{\langle \alpha^* \rangle P \rightarrow Q}$$



differential game logic

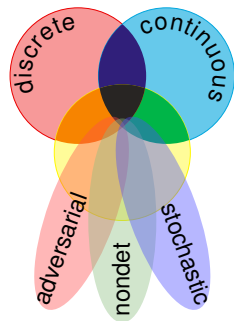
$$\text{dGL} = \text{GL} + \text{HG} = \text{dL} + \text{d}$$



- Axiomatics for hybrid games
- Proving winning strategies

Next chapter

- 1 Soundness
- 2 Proofs
- 3 Separations





André Platzer.

Logical Foundations of Cyber-Physical Systems.

Springer, Switzerland, 2018.

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