Operators of Differential Dynamic Logic (dL)

dL	Operator	Meaning
$e \geq \tilde{e}$	greater or equals	true if value of e greater-or-equal to $\tilde{e}$
$\neg P$	negation / not	true if P is false
$P \land Q$	conjunction / and	true if both P and Q are true
$P \lor Q$	disjunction / or	true if P is true or if Q is true
$P \to Q$	implication / implies	true if P is false or Q is true
$P \leftrightarrow Q$	bi-implication / equivalent	true if P and Q are both true or both false
$\forall x\, P$	universal quantifier / for all	true if P is true for all values of variable x
$\exists x\, P$	existential quantifier / exist	true if P is true for some value of variable x
$[\alpha]P$	$[\cdot]$ modality / box	true if P is true after all runs of HP α
$\langle\alpha\rangle P$	$\langle\cdot\rangle$ modality / diamond	true if P is true after some run of HP α

Statements and effects of Hybrid Programs (HPs)

HP Notation	Operation	Effect
$x := e$	discrete assignment	assigns current value of term e to variable x
$x := *$	nondet. assignment	assigns any real value to variable x
$x' = f(x) \,\&\, Q$	continuous evolution	follow differential equation $x' = f(x)$ within evolution domain Q for any duration
$?Q$	state test / check	test first-order formula Q at current state
$\alpha; \beta$	seq. composition	HP β starts after HP α finishes
$\alpha \cup \beta$	nondet. choice	choice between alternatives HP α or HP β
α^*	nondet. repetition	repeats HP α any $n \in \mathbb{N}$ times

Semantics of dL formula P is the set of states $[\![P]\!] \subseteq \mathscr{S}$ in which it is true

$[\![e \geq \tilde{e}]\!] = \{\omega \in \mathscr{S} : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\}$

$[\![P \land Q]\!] = [\![P]\!] \cap [\![Q]\!]$

$[\![P \lor Q]\!] = [\![P]\!] \cup [\![Q]\!]$

$[\![\neg P]\!] = [\![P]\!]^{\complement} = \mathscr{S} \setminus [\![P]\!]$

$[\![\langle\alpha\rangle P]\!] = [\![\alpha]\!] \circ [\![P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for some state } \nu \text{ such that } (\omega,\nu) \in [\![\alpha]\!]\}$

$[\![[\alpha]P]\!] = [\![\neg\langle\alpha\rangle\neg P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all states } \nu \text{ such that } (\omega,\nu) \in [\![\alpha]\!]\}$

$[\![\exists x\, P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for some state } \nu \text{ that agrees with } \omega \text{ except on } x\}$

$[\![\forall x\, P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all states } \nu \text{ that agree with } \omega \text{ except on } x\}$

Semantics of HP α is relation $[\![\alpha]\!] \subseteq \mathscr{S} \times \mathscr{S}$ between initial and final states

$[\![x := e]\!] = \{(\omega,\nu) : \nu = \omega \text{ except that } \nu[\![x]\!] = \omega[\![e]\!]\}$

$[\![?Q]\!] = \{(\omega,\omega) : \omega \in [\![Q]\!]\}$

$[\![x' = f(x) \,\&\, Q]\!] = \big\{(\omega,\nu) : \varphi(0) = \omega \text{ except at } x' \text{ and } \varphi(r) = \nu \text{ for a solution } \varphi{:}[0,r] \to \mathscr{S} \text{ of any duration } r \text{ satisfying } \varphi \models x' = f(x) \land Q\big\}$

$[\![\alpha \cup \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$

$[\![\alpha;\beta]\!] = [\![\alpha]\!] \circ [\![\beta]\!] = \{(\omega,\nu) : (\omega,\mu) \in [\![\alpha]\!], (\mu,\nu) \in [\![\beta]\!]\}$

$[\![\alpha^*]\!] = [\![\alpha]\!]^* = \bigcup_{n\in\mathbb{N}} [\![\alpha^n]\!] \text{ with } \alpha^{n+1} \equiv \alpha^n;\alpha \text{ and } \alpha^0 \equiv ?true$

Axiomatization (dL)

$\langle\cdot\rangle$ $\langle\alpha\rangle P \leftrightarrow \neg[\alpha]\neg P$

$[:=]$ $[x:=e]p(x) \leftrightarrow p(e)$

$[?]$ $[?Q]P \leftrightarrow (Q \rightarrow P)$

$[']$ $[x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0\,[x:=y(t)]p(x) \quad (y'(t) = f(y))$

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

$[^*]$ $[\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$

K $[\alpha](P \rightarrow Q) \rightarrow ([\alpha]P \rightarrow [\alpha]Q)$

I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$

V $p \rightarrow [\alpha]p \qquad (FV(p) \cap BV(\alpha) = \emptyset)$

M$[\cdot]$ $\dfrac{P \rightarrow Q}{[\alpha]P \rightarrow [\alpha]Q}$

G $\dfrac{P}{[\alpha]P}$

Differential equation axioms

DW $[x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q](Q \rightarrow P)$

DI $\big([x' = f(x)\,\&\,Q]P \leftrightarrow [?Q]P\big) \leftarrow (Q \rightarrow [x' = f(x)\,\&\,Q](P)')$

DC $\big([x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q \wedge C]P\big) \leftarrow [x' = f(x)\,\&\,Q]C$

DE $[x' = f(x)\,\&\,Q]P \leftrightarrow [x' = f(x)\,\&\,Q][x' := f(x)]P$

DG $[x' = f(x)\,\&\,Q]P \leftrightarrow \exists y\,[x' = f(x), y' = a(x)\cdot y + b(x)\,\&\,Q]P$

$+'$ $(e+k)' = (e)' + (k)'$

$\cdot'$ $(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$

c' $(c())' = 0 \qquad$ (for numbers or constants $c()$)

x' $(x)' = x' \qquad$ (for variable $x \in \mathscr{V}$)

Differential equation proof rules

dW $\dfrac{Q \vdash P}{\Gamma \vdash [x' = f(x)\,\&\,Q]P, \Delta}$

dI $\dfrac{Q \vdash [x':=f(x)](F)'}{F \vdash [x' = f(x)\,\&\,Q]F}$

dC $\dfrac{\Gamma \vdash [x' = f(x)\,\&\,Q]C, \Delta \qquad \Gamma \vdash [x' = f(x)\,\&\,(Q \wedge C)]P, \Delta}{\Gamma \vdash [x' = f(x)\,\&\,Q]P, \Delta}$

Sequent calculus proof rules

¬R $\dfrac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta}$ ∧R $\dfrac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \land Q, \Delta}$ ∨R $\dfrac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \lor Q, \Delta}$

¬L $\dfrac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta}$ ∧L $\dfrac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \land Q \vdash \Delta}$ ∨L $\dfrac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \lor Q \vdash \Delta}$

→R $\dfrac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \to Q, \Delta}$ id $\dfrac{}{\Gamma, P \vdash P, \Delta}$ WR $\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash P, \Delta}$

→L $\dfrac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \to Q \vdash \Delta}$ cut $\dfrac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}$ WL $\dfrac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}$

∀R $\dfrac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}$ $(y \notin \Gamma, \Delta, \forall x\, p(x))$ ∃R $\dfrac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}$ (arbitrary term e)

∀L $\dfrac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}$ (arbitrary term e) ∃L $\dfrac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}$ $(y \notin \Gamma, \Delta, \exists x\, p(x))$

CER $\dfrac{\Gamma \vdash C(Q), \Delta \quad \vdash P \leftrightarrow Q}{\Gamma \vdash C(P), \Delta}$ =R $\dfrac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta}$

CEL $\dfrac{\Gamma, C(Q) \vdash \Delta \quad \vdash P \leftrightarrow Q}{\Gamma, C(P) \vdash \Delta}$ =L $\dfrac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$

Derived axioms and derived rules

∧′ $(P \land Q)' \leftrightarrow (P)' \land (Q)'$

∨′ $(P \lor Q)' \leftrightarrow (P)' \land (Q)'$

[]∧ $[\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q$

$\overleftarrow{[*]}$ $[\alpha^*]P \leftrightarrow P \land [\alpha^*][\alpha]P$

[∗∗] $[\alpha^*; \alpha^*]P \leftrightarrow [\alpha^*]P$

$[:=]_=$ $\dfrac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}$ (y new)

iG $\dfrac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}$ (y new)

dG $\dfrac{\Gamma \vdash \exists y\, [x' = f(x), y' = a(x) \cdot y + b(x)\, \&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x)\, \&\, Q]P, \Delta}$

dA $\dfrac{\vdash J \leftrightarrow \exists y\, G \quad G \vdash [x' = f(x), y' = a(x) \cdot y + b(x)\, \&\, Q]G}{J \vdash [x' = f(x)\, \&\, Q]J}$