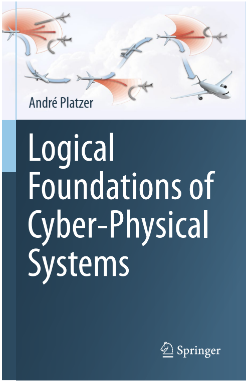
12: Ghosts & Differential Ghosts

Logical Foundations of Cyber-Physical Systems



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Outline

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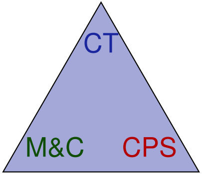
Learning Objectives

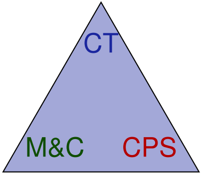
rigorous reasoning about ODEs
extra dimensions for extra invariants
invent dark energy
intuition for differential invariants
states and proofs
verify CPS models at scale



none: ghosts are for proofs!

relations of state
extra ghost state
CPS semantics

rigorous reasoning about ODEs
extra dimensions for extra invariants
invent dark energy
intuition for differential invariants
states and proofs
verify CPS models at scale



mark ghosts in models
syntax of models
solutions of ODEs

relations of state
extra ghost state
CPS semantics

Outline

Differential Weakening

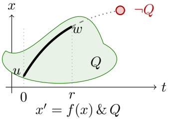
$$\frac{Q \vdash F}{P \vdash [x' = f(x) \,\&\, Q]F}$$

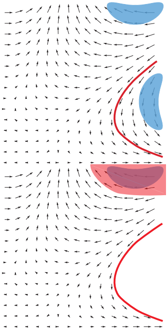
Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$





DW $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q](Q \to F)$

DI $[x' = f(x) \,\&\, Q]F \leftarrow \big(Q \to F \land [x' = f(x) \,\&\, Q](F)'\big)$

DC $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q \land C]F\big) \leftarrow [x' = f(x) \,\&\, Q]C$

Differential Invariants for Differential Equations

Differential Weakening

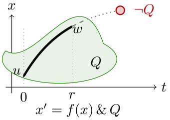
$$\frac{Q \vdash F}{P \vdash [x' = f(x) \,\&\, Q]F}$$

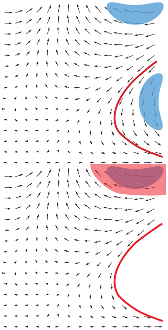
Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \quad F \vdash [x' = f(x) \,\&\, Q \land C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$





DW $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow F)$

DI $[x' = f(x) \,\&\, Q]F \leftarrow \big(Q \rightarrow F \land [x' = f(x) \,\&\, Q](F)'\big)$

DC $\big([x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q \land C]F\big) \leftarrow [x' = f(x) \,\&\, Q]C$

DE $[x' = f(x) \,\&\, Q]F \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]F$

Outline

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\quad (y \text{ new})$$

$$\to\text{R}\ \frac{}{\vdash xy - 1 = 0 \to [x' = x, y' = -y]xy = 1}$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta} \quad (y \text{ new})$$

$$\text{iG}\ \frac{}{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1}$$

$$\rightarrow\text{R}\ \frac{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1}{\vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1}$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG} \quad \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta} \quad (y \text{ new})$$

discrete ghost c remembers function of old state

$$\dfrac{\dfrac{\dfrac{}{xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1}\,[:=]_=}{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1}\,\text{iG}}{\vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1}\,\rightarrow\text{R}$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new})$$

$p \leftrightarrow [y := e]p$ by $[:=]$

$$[:=]_{=}\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}$$

$$\begin{array}{ll}
\text{MR} & \overline{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1} \\
{[:=]_{=}} & \overline{xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1} \\
\text{iG} & \overline{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1} \\
\rightarrow\text{R} & \overline{\vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1}
\end{array}$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new}) \qquad p \leftrightarrow [y := e]p \text{ by } [:=]$$

$$[:=]_=\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$\begin{array}{ll}
\text{dI} & \overline{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy} \quad \triangleright \\
\text{MR} & \overline{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1} \\
[:=]_= & \overline{xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1} \\
\text{iG} & \overline{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1} \\
\rightarrow\text{R} & \overline{\vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1}
\end{array}$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y\ \text{new})$$

$p \leftrightarrow [y := e]p$ by $[:=]$

$$[:=]_= \ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y\ \text{new})$$

$$\begin{array}{ll}
[:=] & \overline{\vdash [x' := x][y' := -y]0 = x'y + xy'} \\
\text{dI} & \overline{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy \qquad \triangleright} \\
\text{MR} & \overline{xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1} \\
[:=]_= & \overline{xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1} \\
\text{iG} & \overline{xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1} \\
\rightarrow\text{R} & \overline{\vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1}
\end{array}$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new}) \qquad p \leftrightarrow [y := e]p \text{ by } [:=]$$

$$[:=]_= \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$
\begin{array}{ll}
\mathbb{R} & \vdash 0 = xy + x(-y) \\
\hline
[:=] & \vdash [x':=x][y':=-y]0 = x'y + xy' \\
\hline
\text{dI} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy \qquad \triangleright \\
\hline
\text{MR} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1 \\
\hline
[:=]_= & xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1 \\
\hline
\text{iG} & xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1 \\
\hline
\rightarrow\text{R} & \vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1
\end{array}
$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

Discrete Ghosts

$$\text{iG}\ \frac{\Gamma \vdash [y := e]p, \Delta}{\Gamma \vdash p, \Delta}\ (y \text{ new})$$

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$$[:=]_=\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}(y \text{ new})$$

$$
\begin{array}{ll}
 & * \\
\mathbb{R} & \vdash 0 = xy + x(-y) \\
[:=] & \vdash [x' := x][y' := -y]0 = x'y + xy' \\
\text{dI} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]c = xy \qquad \triangleright \\
\text{MR} & xy - 1 = 0, c = xy \vdash [x' = x, y' = -y]xy = 1 \\
[:=]_= & xy - 1 = 0 \vdash [c := xy][x' = x, y' = -y]xy = 1 \\
\text{iG} & xy - 1 = 0 \vdash [x' = x, y' = -y]xy = 1 \\
\rightarrow\text{R} & \vdash xy - 1 = 0 \rightarrow [x' = x, y' = -y]xy = 1
\end{array}
$$

Clou: Ask a ghost to remember some auxiliary state for the proof.

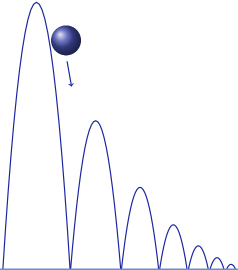
Quantum's Back for a Differential Invariant Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{x\geq0 \vdash 2gv = -2v(-g)}\,\mathbb{R}}{x\geq0 \vdash [x':=v][v':=-g]2gx' = -2vv'}\,[:=]}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq0]2gx=2gH-v^2}\,\text{dI} \qquad \dfrac{\dfrac{*}{x\geq0 \vdash x\geq0}\,\text{id}}{\vdash [x''=-g\,\&\,x\geq0]x\geq0}\,\text{dW}}{2gx=2gH-v^2 \vdash [x''=-g\,\&\,x\geq0](2gx=2gH-v^2 \wedge x\geq0)}\,[]\wedge$$

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.



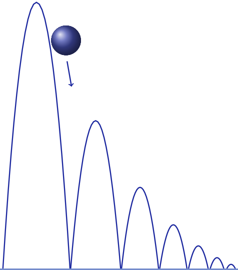
$$
{\scriptstyle []\wedge}\dfrac{
{\scriptstyle \mathrm{dI}}\dfrac{
{\scriptstyle [:=]}\dfrac{
{\scriptstyle \mathbb{R}}\dfrac{*}{x\geq 0 \vdash 2gv = -2v(-g)}
}{x\geq 0 \vdash [x':=v][v':=-g]2gx' = -2vv'}
}{2gx=2gH-v^2 \vdash [x''=-g \,\&\, x\geq 0]2gx=2gH-v^2}
\quad
{\scriptstyle \mathrm{dW}}\dfrac{
{\scriptstyle \mathrm{id}}\dfrac{*}{x\geq 0 \vdash x\geq 0}
}{\vdash [x''=-g \,\&\, x\geq 0]x\geq 0}
}{2gx=2gH-v^2 \vdash [x''=-g \,\&\, x\geq 0](2gx=2gH-v^2 \wedge x\geq 0)}
$$

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.

But need to have the right invariant.

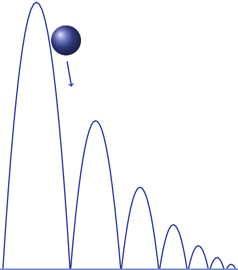


$$A \vdash [x'' = -g \,\&\, x \geq 0]B(x,v)$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Solution:

$$x =$$

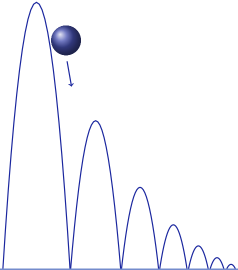
$$v =$$

$$A \vdash [x'' = -g \,\&\, x \geq 0] B(x,v)$$

$$A \equiv$$

$$B(x,v) \equiv$$

$$x'' = -g \equiv \{x' = v, v' = -g\}$$

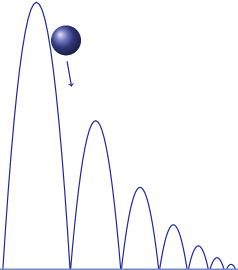


Quantum's Back for a Solution Proof w/o Solution Axiom

Solution:

$$x(t) = x + vt - \frac{g}{2}t^2$$
$$v(t) = v - gt$$

$$A \vdash [x'' = -g \,\&\, x \geq 0]B(x,v)$$
$$A \equiv \text{ redacted}$$
$$B(x,v) \equiv \text{ redacted}$$
$$x'' = -g \equiv \{x' = v, v' = -g\}$$

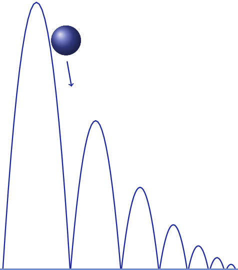


Quantum's Back for a Solution Proof w/o Solution Axiom

Solution: How to use a solution without really trying solution axiom [′]

$$x(t) = x + vt - \frac{g}{2}t^2$$
$$v(t) = v - gt$$

$$A \vdash [x'' = -g \,\&\, x \geq 0]B(x,v)$$
$$A \equiv \text{ redacted}$$
$$B(x,v) \equiv \text{ redacted}$$
$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Quantum's Back for a Solution Proof w/o Solution Axiom

Solution: How to use a solution without really trying solution axiom [′]

$$x(t) = x + vt - \frac{g}{2}t^2$$

$$v(t) = v - gt$$

solution of ODE invariant along ODE

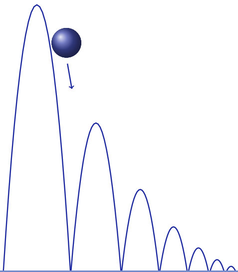
Can't just say $x(t)$

$$A \vdash [x'' = -g \,\&\, x \geq 0] B(x,v)$$

$$A \equiv \text{ redacted}$$

$$B(x,v) \equiv \text{ redacted}$$

$$x'' = -g \equiv \{x' = v, v' = -g\}$$

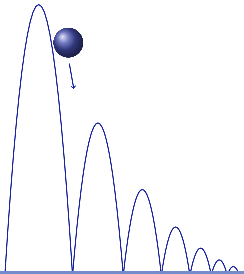


Quantum's Back for a Solution Proof w/o Solution Axiom

Solution: How to use a solution without really trying solution axiom [′]

$$x \quad = x + vt - \frac{g}{2}t^2$$
$$v \quad = v - gt$$

$$A \vdash [x'' = -g \,\&\, x \geq 0]B(x,v)$$
$$A \equiv \text{ redacted}$$
$$B(x,v) \equiv \text{ redacted}$$
$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Quantum's Back for a Solution Proof w/o Solution Axiom

Solution: How to use a solution without really trying solution axiom [′]

$$\begin{aligned} x &= x_0 + v_0 t - \tfrac{g}{2} t^2 \\ v &= v_0 - gt \end{aligned}$$

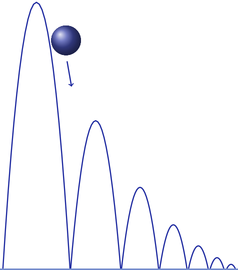
initial velocity v_0 before ODE

$$A \vdash [x'' = -g \,\&\, x \geq 0] B(x,v)$$

$A \equiv$ redacted

$B(x,v) \equiv$ redacted

$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Quantum's Back for a Solution Proof w/o Solution Axiom

Solution: How to use a solution without really trying solution axiom [′]

$$x = x_0 + v_0 t - \frac{g}{2}t^2$$

$$v = v_0 - gt$$

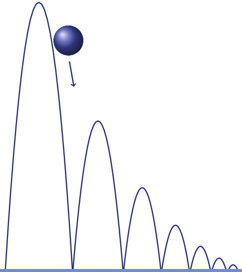
initial velocity v_0 before ODE How?

$$A \vdash [x'' = -g \,\&\, x \geq 0]B(x,v)$$

$A \equiv$ redacted

$B(x,v) \equiv$ redacted

$$x'' = -g \equiv \{x' = v, v' = -g\}$$



Proving Bouncing Balls with Sneaky Solutions

$$\text{iG}\ \frac{}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

Proving Bouncing Balls with Sneaky Solutions

$$\dfrac{\text{dC}\quad \overline{\quad A \vdash [v_0 := v][x''=-g, t'=1 \,\&\, x \geq 0]B(x,v)\quad}}{\text{iG}\quad A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

$$\text{dI}\ \dfrac{}{A \vdash [v_0 := v][x''=-g, t'=1 \,\&\, x \geq 0]\, v = v_0 - tg}$$

$$\text{iG}\ \dfrac{\text{dC}\ \dfrac{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}{A \vdash [v_0 := v][x''=-g, t'=1 \,\&\, x \geq 0]B(x,v)}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

$$\dfrac{[:=]\;\dfrac{}{x \geq 0 \vdash [v' := -g][t' := 1]v' = -t'g}}{\text{dI}\quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]\, v = v_0 - tg}$$

$$\dfrac{\text{iG}\;\dfrac{}{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg]B(x,v)}}{\text{dC}\quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

$$\text{iG}\quad \dfrac{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

$$
\dfrac{\dfrac{\dfrac{}{x \geq 0 \vdash -g = -1g}\,\mathbb{R}}{x \geq 0 \vdash [v' {:=} {-g}][t' {:=} 1]v' = -t'g}\,[:=]}{A \vdash [v_0 := v][x''{=}{-g}, t'{=}1 \,\&\, x{\geq}0]v = v_0 - tg}\,\text{dI}
$$

$$
\dfrac{\dfrac{\dfrac{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}{}\,\text{iG}}{A \vdash [v_0 := v][x''{=}{-g}, t'{=}1 \,\&\, x \geq 0]B(x,v)}\,\text{dC}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}\,\text{iG}
$$

$$\dfrac{\dfrac{\dfrac{*}{x \geq 0 \vdash -g = -1g}\ \mathbb{R}}{x \geq 0 \vdash [v' {:=} -g][t' {:=} 1]v' = -t'g}\ [:=]}{A \vdash [v_0 := v][x''=-g, t'=1 \,\&\, x \geq 0]v = v_0 - tg}\ \text{dI}$$

$$\dfrac{\dfrac{\dfrac{\triangleleft}{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}\ \text{iG}}{A \vdash [v_0 := v][x''=-g, t'=1 \,\&\, x \geq 0]B(x,v)}\ \text{dC}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}\ \text{iG}$$

$$\dfrac{\dfrac{\dfrac{*}{x \geq 0 \vdash -g = -1g}\ \mathbb{R}}{x \geq 0 \vdash [v' := -g][t' := 1]v' = -t'g}\ [:=]}{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]v = v_0 - tg}\ \mathrm{dI}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}\ \mathrm{dC}}{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}\ \mathrm{iG}}{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}\ \mathrm{dC}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}\ \mathrm{iG}$$

$$A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2] B(x,v)$$

dI $\quad A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] x = x_0 + v_0 t - \frac{g}{2} t^2 \;\triangleright$

dC $\quad A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v)$

iG $\quad \triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v)$

dC $\quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v)$

iG $\quad A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v)$

$$A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2]B(x,v)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[:=]\;\; x\geq 0 \land v = v_0 - tg \vdash [x' := v][t' := 1]x' = v_0 t' - 2\frac{g}{2} t t'}{A \vdash [x_0 := x][v_0 := v][x''=-g, t'=1 \,\&\, x\geq 0 \land v = v_0 - tg]x = x_0 + v_0 t - \frac{g}{2}t^2 \;\triangleright}\text{dI}}{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}\text{dC}}{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}\text{iG}}{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}\text{dC}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}\text{iG}$$

$$\overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2] B(x,v)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x{\geq}0 \land v{=}v_0{-}tg \vdash v = v_0 - 2\frac{g}{2} t}\text{id}}{x{\geq}0 \land v{=}v_0{-}tg \vdash [x'{:=}v][t'{:=}1] x' = v_0 t' - 2\frac{g}{2} t t'}[{:=}]}{A \vdash [x_0 := x][v_0 := v][x''{=}{-}g, t'{=}1 \,\&\, x{\geq}0 \land v{=}v_0{-}tg] x{=}x_0{+}v_0 t{-}\frac{g}{2} t^2 \triangleright}\text{dI}}{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x{\geq}0 \land v{=}v_0{-}tg] B(x,v)}\text{dC}}{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v)}\text{iG}}{A \vdash [v_0 := v][x''{=}{-}g, t'{=}1 \,\&\, x \geq 0] B(x,v)}\text{dC}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v)}\text{iG}$$

$$A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2} t^2] B(x,v)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{x \geq 0 \land v = v_0 - tg \vdash v = v_0 - 2\frac{g}{2} t}\text{id}}{x \geq 0 \land v = v_0 - tg \vdash [x' := v][t' := 1] x' = v_0 t' - 2\frac{g}{2} t t'}[:=]}{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] x = x_0 + v_0 t - \frac{g}{2} t^2 \triangleright}\text{dI}}{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v)}\text{dC}}{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg] B(x,v)}\text{iG}}{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v)}\text{dC}}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0] B(x,v)}\text{iG}$$

$$\text{dW}\ \dfrac{x \geq 0 \wedge v = v_0 - tg \wedge x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash B(x,v)}{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg \wedge x = x_0 + v_0 t - \frac{g}{2}t^2]B(x,v)}$$

$$*$$

$$\text{id}\ \overline{x \geq 0 \wedge v = v_0 - tg \vdash v = v_0 - 2\frac{g}{2}t}$$

$$[:=]\ \overline{x \geq 0 \wedge v = v_0 - tg \vdash [x' := v][t' := 1]x' = v_0 t' - 2\frac{g}{2}tt'}$$

$$\text{dI}\ \overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg]x = x_0 + v_0 t - \frac{g}{2}t^2 \triangleright}$$

$$\text{dC}\ \overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg]B(x,v)}$$

$$\text{iG}\ \overline{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg]B(x,v)}$$

$$\text{dC}\ \overline{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

$$\text{iG}\ \overline{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

$$\dfrac{\wedge L\;\dfrac{}{x \geq 0 \wedge v = v_0 - tg \wedge x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash 2gx = 2gH - v^2 \wedge x \geq 0}}{x \geq 0 \wedge v = v_0 - tg \wedge x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash B(x,v)}$$

$$\text{dW}\;\dfrac{x \geq 0 \wedge v = v_0 - tg \wedge x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash B(x,v)}{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg \wedge x = x_0 + v_0 t - \frac{g}{2}t^2]B(x,v)}$$

$$*$$

$$\text{id}\;\overline{x \geq 0 \wedge v = v_0 - tg \vdash v = v_0 - 2\tfrac{g}{2}t}$$

$$[:=]\;\overline{x \geq 0 \wedge v = v_0 - tg \vdash [x' := v][t' := 1]x' = v_0 t' - 2\tfrac{g}{2}tt'}$$

$$\text{dI}\;\overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg]x = x_0 + v_0 t - \tfrac{g}{2}t^2} \;\triangleright$$

$$\text{dC}\;\overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg]B(x,v)}$$

$$\text{iG}\;\overline{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \wedge v = v_0 - tg]B(x,v)}$$

$$\text{dC}\;\overline{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

$$\text{iG}\;\overline{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

$$\cfrac{\cfrac{x \geq 0, v = v_0 - tg, x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0}{x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0}\,\text{=R}}{\cfrac{x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash B(x,v)}{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2}t^2]B(x,v)}\,\text{dW}}\,\land\text{L}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{*}{x \geq 0 \land v = v_0 - tg \vdash v = v_0 - 2\frac{g}{2}t}\,\text{id}}{x \geq 0 \land v = v_0 - tg \vdash [x' := v][t' := 1]x' = v_0 t' - 2\frac{g}{2}tt'}\,[:=]}{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]x = x_0 + v_0 t - \frac{g}{2}t^2 \,\triangleright}\,\text{dI}}{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}\,\text{dC}}{\triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}\,\text{iG}}{\cfrac{A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}{A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}\,\text{iG}}\,\text{dC}$$

$$
\begin{array}{l}
=\!R\quad x \geq 0, v = v_0 - tg, x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash 2gx = 2gH - (v_0 - tg)^2 \land x \geq 0 \\
=\!R\quad x \geq 0, v = v_0 - tg, x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0 \\
\land L\quad x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0 \\
\quad x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash B(x,v) \\
dW\quad A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2}t^2]B(x,v)
\end{array}
$$

$$
\begin{array}{l}
\quad * \\
id\quad x \geq 0 \land v = v_0 - tg \vdash v = v_0 - 2\frac{g}{2}t \\
[:=]\quad x \geq 0 \land v = v_0 - tg \vdash [x' := v][t' := 1]x' = v_0 t' - 2\frac{g}{2}tt' \\
dI\quad A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]x = x_0 + v_0 t - \frac{g}{2}t^2 \triangleright \\
dC\quad A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v) \\
iG\quad \triangleleft \quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v) \\
dC\quad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v) \\
iG\quad A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)
\end{array}
$$

$$\text{WL}\ \overline{\qquad x \geq 0, v = v_0 - tg, x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash 2g(x_0 + v_0 t - \frac{g}{2}t^2) = 2gH - (v_0 - tg)^2 \land x \geq 0}$$

$$\text{=R}\ \overline{\qquad x \geq 0, v = v_0 - tg, x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash 2gx = 2gH - (v_0 - tg)^2 \land x \geq 0}$$

$$\text{=R}\ \overline{\qquad x \geq 0, v = v_0 - tg, x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0}$$

$$\land\text{L}\ \overline{\qquad x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash 2gx = 2gH - v^2 \land x \geq 0}$$

$$\overline{\qquad x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2}t^2 \vdash B(x,v)}$$

$$\text{dW}\ \overline{A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg \land x = x_0 + v_0 t - \frac{g}{2}t^2]B(x,v)}$$

$$*$$

$$\text{id}\ \overline{x \geq 0 \land v = v_0 - tg \vdash v = v_0 - 2\frac{g}{2}t}$$

$$[:=]\ \overline{x \geq 0 \land v = v_0 - tg \vdash [x' := v][t' := 1]x' = v_0 t' - 2\frac{g}{2}tt'}$$

$$\text{dI}\ \overline{\qquad A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]x = x_0 + v_0 t - \frac{g}{2}t^2 \triangleright}$$

$$\text{dC}\ \overline{\qquad A \vdash [x_0 := x][v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}$$

$$\text{iG}\ \overline{\triangleleft \qquad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0 \land v = v_0 - tg]B(x,v)}$$

$$\text{dC}\ \overline{\qquad A \vdash [v_0 := v][x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

$$\text{iG}\ \overline{\qquad A \vdash [x'' = -g, t' = 1 \,\&\, x \geq 0]B(x,v)}$$

$$\wedge R \; \dfrac{}{x \geq 0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \wedge x\geq 0}$$

$$WL \; \dfrac{}{x \geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \wedge x\geq 0}$$

$$=R \; \dfrac{}{x \geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-(v_0-tg)^2 \wedge x\geq 0}$$

$$=R \; \dfrac{}{x \geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2 \wedge x\geq 0}$$

$$\wedge L \; \dfrac{}{x \geq 0 \wedge v=v_0-tg \wedge x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2 \wedge x\geq 0}$$

$$\dfrac{}{x \geq 0 \wedge v=v_0-tg \wedge x=x_0+v_0t-\frac{g}{2}t^2 \vdash B(x,v)}$$

$$dW \; \dfrac{}{A \vdash [x_0:=x][v_0:=v][x''=-g,t'=1 \,\&\, x\geq 0 \wedge v=v_0-tg \wedge x=x_0+v_0t-\frac{g}{2}t^2]B(x,v)}$$

$$*$$

$$id \; \dfrac{}{x\geq 0 \wedge v=v_0-tg \vdash v=v_0-2\frac{g}{2}t}$$

$$[:=] \; \dfrac{}{x\geq 0 \wedge v=v_0-tg \vdash [x':=v][t':=1]x'=v_0t'-2\frac{g}{2}tt'}$$

$$dI \; \dfrac{}{A \vdash [x_0:=x][v_0:=v][x''=-g,t'=1 \,\&\, x\geq 0 \wedge v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \rhd}$$

$$dC \; \dfrac{}{A \vdash [x_0:=x][v_0:=v][x''=-g,t'=1 \,\&\, x\geq 0 \wedge v=v_0-tg]B(x,v)}$$

$$iG \; \dfrac{\lhd}{A \vdash [v_0:=v][x''=-g,t'=1 \,\&\, x\geq 0 \wedge v=v_0-tg]B(x,v)}$$

$$dC \; \dfrac{}{A \vdash [v_0:=v][x''=-g,t'=1 \,\&\, x\geq 0]B(x,v)}$$

$$iG \; \dfrac{}{A \vdash [x''=-g,t'=1 \,\&\, x\geq 0]B(x,v)}$$

Proving Bouncing Balls with Sneaky Solutions

WL $$x\geq 0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2$$ id $$x\geq 0 \vdash x\geq 0$$

∧R $$x\geq 0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq 0$$

WL $$x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq 0$$

=R $$x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-(v_0-tg)^2\land x\geq 0$$

=R $$x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2\land x\geq 0$$

∧L $$x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2\land x\geq 0$$

$$x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2 \vdash B(x,v)$$

dW $$A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2]B(x,v)$$

$$*$$

id $$x\geq 0\land v=v_0-tg\vdash v=v_0-2\frac{g}{2}t$$

[:=] $$x\geq 0\land v=v_0-tg\vdash [x':=v][t':=1]x'=v_0t'-2\frac{g}{2}tt'$$

dI $$A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \triangleright$$

dC $$A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg]B(x,v)$$

iG $$\triangleleft \quad A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\land v=v_0-tg]B(x,v)$$

dC $$A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0]B(x,v)$$

iG $$A\vdash [x''=-g,t'=1\,\&\,x\geq 0]B(x,v)$$

Proving Bouncing Balls with Sneaky Solutions

$$\vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2$$

$$\text{WL}\ \ x\geq0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \text{id}\ \ x\geq0\vdash x\geq0$$

$$\land\text{R}\ \ x\geq0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq0$$

$$\text{WL}\ \ x\geq0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\land x\geq0$$

$$=\text{R}\ \ x\geq0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-(v_0-tg)^2\land x\geq0$$

$$=\text{R}\ \ x\geq0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2\land x\geq0$$

$$\land\text{L}\ \ x\geq0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2 \vdash 2gx=2gH-v^2\land x\geq0$$

$$x\geq0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2 \vdash B(x,v)$$

$$\text{dW}\ \ A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\ \&\ x\geq0\land v=v_0-tg\land x=x_0+v_0t-\frac{g}{2}t^2]B(x,v)$$

$$*$$

$$\text{id}\ \ x\geq0\land v=v_0-tg\vdash v=v_0-2\frac{g}{2}t$$

$$[:=]\ \ x\geq0\land v=v_0-tg\vdash [x':=v][t':=1]x'=v_0t'-2\frac{g}{2}tt'$$

$$\text{dI}\ \ A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\ \&\ x\geq0\land v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2\ \triangleright$$

$$\text{dC}\ \ A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\ \&\ x\geq0\land v=v_0-tg]B(x,v)$$

$$\text{iG}\ \ \triangleleft\qquad A\vdash [v_0:=v][x''=-g,t'=1\ \&\ x\geq0\land v=v_0-tg]B(x,v)$$

$$\text{dC}\ \ A\vdash [v_0:=v][x''=-g,t'=1\ \&\ x\geq0]B(x,v)$$

$$\text{iG}\ \ A\vdash [x''=-g,t'=1\ \&\ x\geq0]B(x,v)$$

Proving Bouncing Balls with Sneaky Solutions

$$\vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-v_0^2+2v_0tg-t^2g^2$$

$$\vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2$$

$$\text{WL}\quad x\geq 0\vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \text{id}\quad x\geq 0\vdash x\geq 0$$

$$\wedge\text{R}\quad x\geq 0\vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\wedge x\geq 0$$

$$\text{WL}\quad x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2\vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2\wedge x\geq 0$$

$$=\text{R}\quad x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2\vdash 2gx=2gH-(v_0-tg)^2\wedge x\geq 0$$

$$=\text{R}\quad x\geq 0, v=v_0-tg, x=x_0+v_0t-\frac{g}{2}t^2\vdash 2gx=2gH-v^2\wedge x\geq 0$$

$$\wedge\text{L}\quad x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\frac{g}{2}t^2\vdash 2gx=2gH-v^2\wedge x\geq 0$$

$$x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\frac{g}{2}t^2\vdash B(x,v)$$

$$\text{dW}\quad A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\frac{g}{2}t^2]B(x,v)$$

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$$\text{dI}\quad A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \triangleright$$

$$\text{dC}\quad A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg]B(x,v)$$

$$\text{iG}\quad \triangleleft \quad A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0\wedge v=v_0-tg]B(x,v)$$

$$\text{dC}\quad A\vdash [v_0:=v][x''=-g,t'=1\,\&\,x\geq 0]B(x,v)$$

$$\text{iG}\quad A\vdash [x''=-g,t'=1\,\&\,x\geq 0]B(x,v)$$

Proving Bouncing Balls with Sneaky Solutions

$$\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-v_0^2+2v_0tg-t^2g^2$$

$$\vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2$$

$$\text{WL}\quad x\geq 0 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \text{id}\ \frac{*}{x\geq 0\vdash x\geq 0}$$

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$$x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\tfrac{g}{2}t^2\vdash B(x,v)$$

$$\text{dW}\quad A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\ \&\ x\geq 0\wedge v=v_0-tg\wedge x=x_0+v_0t-\tfrac{g}{2}t^2]B(x,v)$$

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$$\text{id}\quad x\geq 0\wedge v=v_0-tg\vdash v=v_0-2\tfrac{g}{2}t$$

$$[:=]\quad x\geq 0\wedge v=v_0-tg\vdash [x':=v][t':=1]x'=v_0t'-2\tfrac{g}{2}tt'$$

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$$\text{dC}\quad A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\ \&\ x\geq 0\wedge v=v_0-tg]B(x,v)$$

$$\text{iG}\quad \lhd \quad A\vdash [v_0:=v][x''=-g,t'=1\ \&\ x\geq 0\wedge v=v_0-tg]B(x,v)$$

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$$[:=]\quad x\geq 0\wedge v=v_0-tg\vdash [x':=v][t':=1]x'=v_0t'-2\frac{g}{2}tt'$$

Ghost solutions

$$\text{dI}\quad A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\ \&\ x\geq 0\wedge v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \triangleright$$

$$\text{dC}\quad A\vdash [x_0:=x][v_0:=v][x''=-g,t'=1\ \&\ x\geq 0\wedge v=v_0-tg]B(x,v)$$

$$\text{iG}\quad \triangleleft\ A\vdash [v_0:=v][x''=-g,t'=1\ \&\ x\geq 0\wedge v=v_0-tg]B(x,v)$$

Initial ghost

$$\text{dC}\quad A\vdash [v_0:=v][x''=-g,t'=1\ \&\ x\geq 0]B(x,v)$$

$$\text{iG}\quad A\vdash [x''=-g,t'=1\ \&\ x\geq 0]B(x,v)$$

$$\vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-v_0^2+2v_0tg-t^2g^2$$

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WL $$x\geq 0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \qquad \text{id}\ \frac{*}{x\geq 0 \vdash x\geq 0}$$

$\wedge$R $$x\geq 0 \vdash 2g(x_0+v_0t-\frac{g}{2}t^2)=2gH-(v_0-tg)^2 \wedge x\geq 0$$

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Not initially true

dI $$A \vdash [x_0:=x][v_0:=v][x''=-g, t'=1 \,\&\, x\geq 0 \wedge v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \triangleright$$

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$v=v_0-(t-t_0)g$

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WL $$x\geq 0 \vdash 2g(x_0+v_0t-\tfrac{g}{2}t^2)=2gH-(v_0-tg)^2$$

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ghost $[t_0:=t]$ by iG

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What about time?

dI $$A\vdash [x_0:=x][v_0:=v][x''=-g, t'=1\ \&\ x\geq 0\wedge v=v_0-tg]x=x_0+v_0t-\frac{g}{2}t^2 \triangleright$$

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On the Significance of Time

Why does the proof with ghost solutions need $t' = 1$ in the model?

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Could we just add in $t' = 1$ if we need it?

Differential Ghosts of Time

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$$\frac{\Gamma \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

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- × Cannot add $t' = 1$ to $x' = v, v' = t$

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- × Cannot add $t' = 1$ to $x' = v, v' = t$
- ✓ Can add $t' = 1$ to $x' = v, v' = -g$

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What could possibly go wrong?

- × Cannot add $t' = 1$ to $x' = v, t' = 2$
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This is a perfectly harmless proof rule with fresh t.

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- × Can add $t' = 1$ to $x' = v, v' = -g$ unless e.g. postcondition P reads t

This is a perfectly harmless proof rule with fresh t.
But it's too specific and cannot add any other ODEs.

Constructing Differential Ghosts

Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x,y) \,\&\, Q]P$$

Differential Ghosts of Time

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Differential Ghost $[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x, y) \,\&\, Q]P$

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$$\frac{\Gamma \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta} \quad (t \text{ fresh})$$

Get differential ghosts of time by axiom DG, even with clever initial $t = 0$:

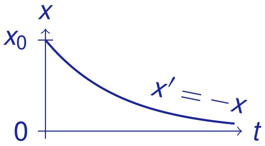
$$\text{DG}\dfrac{\exists\text{R}\dfrac{\Gamma, t = 0 \vdash [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}{\Gamma \vdash \exists t\, [x' = f(x), t' = 1 \,\&\, Q]P, \Delta}}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

Differential Ghost $[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x, y) \,\&\, Q]P$

Exponentials Make Matters Worse

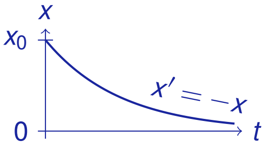
Example ()

$$\text{dI}\;\frac{}{x > 0 \vdash [x' = -x]x > 0}$$



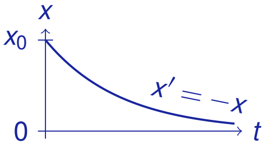
Example ()

$$\dfrac{\dfrac{}{\vdash [x' := -x]x' > 0}\,[:=]}{x > 0 \vdash [x' = -x]x > 0}\,\mathrm{dI}$$



Example ()

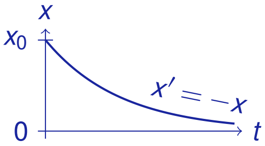
$$\dfrac{\dfrac{\dfrac{}{\vdash -x > 0}\ \mathbb{R}}{\vdash [x' := -x]x' > 0}\ [:=]}{x > 0 \vdash [x' = -x]x > 0}\ \text{dI}$$



Exponentials Make Matters Worse

Example (Cannot prove like this)

$$
\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash -x > 0}\ \mathbb{R}}{\vdash [x' := -x]x' > 0}\ [:=]}{x > 0 \vdash [x' = -x]x > 0}\ \text{dI}
$$

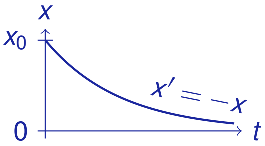


Exponentials Make Matters Worse

Example (Cannot prove like this)

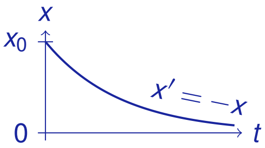
$$
\dfrac{\dfrac{\dfrac{\text{not valid}}{\vdash -x > 0}\,\mathbb{R}}{\vdash [x' := -x]x' > 0}\,[:=]}{x > 0 \vdash [x' = -x]x > 0}\,\mathrm{dI}
$$

Matters get worse over time in this dynamics



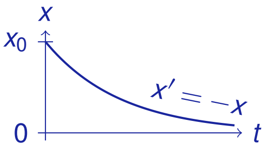
Example (Spooky proof)

$$\text{DG}\ \frac{}{x > 0 \vdash [x' = -x]x > 0}$$



Example (Spooky proof)

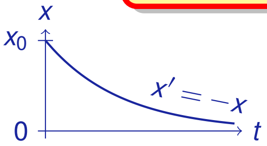
$$\dfrac{\dfrac{}{x > 0 \vdash \exists y\, [x' = -x, y' = \frac{y}{2}]x > 0}\ \exists\text{R,cut}}{x > 0 \vdash [x' = -x]x > 0}\ \text{DG}$$



Example (Spooky proof)

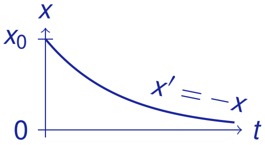
$$\dfrac{\dfrac{\dfrac{}{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}\text{MR}}{x > 0 \vdash \exists y\, [x' = -x, y' = \frac{y}{2}]x > 0}\exists\text{R,cut}}{x > 0 \vdash [x' = -x]x > 0}\text{DG}$$

differential ghost: dream me up



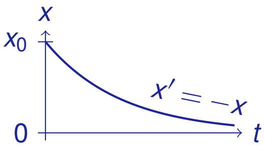
Example (Spooky proof)

$$\dfrac{\dfrac{\dfrac{\mathbb{R}\;\overline{xy^2=1 \vdash x>0} \qquad \text{dI}\;\overline{xy^2=1 \vdash [x' = -x, y' = \frac{y}{2}]xy^2 = 1}}{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}\text{MR}}{x > 0 \vdash \exists y\,[x' = -x, y' = \frac{y}{2}]x > 0}\exists\text{R,cut}}{x > 0 \vdash [x' = -x]x > 0}\text{DG}$$



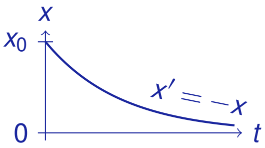
Example (Spooky proof)

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{xy^2{=}1 \vdash x{>}0}\,\mathbb{R} \qquad \dfrac{}{xy^2{=}1 \vdash [x' = -x, y' = \frac{y}{2}]xy^2 = 1}\,\text{dI}}{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}\,\text{MR}}{x > 0 \vdash \exists y\,[x' = -x, y' = \frac{y}{2}]x > 0}\,\exists\text{R,cut}}{x > 0 \vdash [x' = -x]x > 0}\,\text{DG}$$



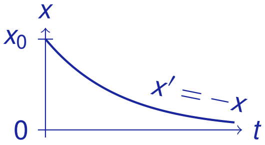
Example (Spooky proof)

$$
\dfrac{\dfrac{\dfrac{*}{xy^2=1 \vdash x>0}\,\mathbb{R} \qquad \dfrac{\dfrac{}{\vdash [x':=-x][y':=\frac{y}{2}]x'y^2+x2yy'=0}\,[:=]}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]xy^2=1}\,\mathrm{dI}}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]x>0}\,\mathrm{MR}}{\dfrac{x>0 \vdash \exists y\,[x'=-x, y'=\frac{y}{2}]x>0}{x>0 \vdash [x'=-x]x>0}\,\mathrm{DG}}\,\exists\mathrm{R,cut}
$$



Example (Spooky proof)

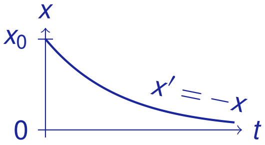
$$
\dfrac{\dfrac{\overset{\ast}{\overline{xy^2=1 \vdash x>0}}\;\mathbb{R} \qquad \dfrac{\dfrac{\dfrac{}{\vdash -x y^2 + 2xy\frac{y}{2} = 0}\;\mathbb{R}}{\vdash [x':=-x][y':=\frac{y}{2}]x'y^2+x2yy'=0}\;[:=]}{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]xy^2=1}\;\mathrm{dI}}{\dfrac{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]x>0}{\dfrac{x>0 \vdash \exists y\,[x'=-x, y'=\frac{y}{2}]x>0}{x>0 \vdash [x'=-x]x>0}\;\mathrm{DG}}\;\exists\mathrm{R,cut}}\;\mathrm{MR}
$$



Example (Spooky proof)

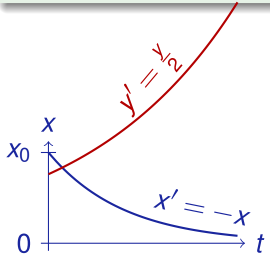
$$
\dfrac{\mathbb{R}\,\dfrac{*}{xy^2{=}1 \vdash x{>}0} \qquad \text{dI}\,\dfrac{[:=]\,\dfrac{\mathbb{R}\,\dfrac{*}{\vdash -xy^2 + 2xy\frac{y}{2} = 0}}{\vdash [x'{:=}{-x}][y'{:=}\frac{y}{2}]x'y^2{+}x2yy'{=}0}}{xy^2{=}1 \vdash [x' = -x, y' = \frac{y}{2}]xy^2 = 1}}{\text{MR}\;\; xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}
$$

$$
\exists\text{R,cut}\;\; \dfrac{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}{x > 0 \vdash \exists y\, [x' = -x, y' = \frac{y}{2}]x > 0}
\qquad
\text{DG}\;\; \dfrac{x > 0 \vdash \exists y\, [x' = -x, y' = \frac{y}{2}]x > 0}{x > 0 \vdash [x' = -x]x > 0}
$$



Example (Spooky proof with counterweight ghost)

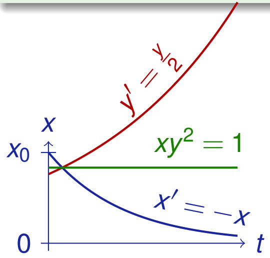
$$\dfrac{\mathbb{R}\;\dfrac{*}{xy^2=1 \vdash x>0} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x':=-x][y':=\frac{y}{2}]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]xy^2=1}}{\dfrac{\text{MR}\quad xy^2=1 \vdash [x'=-x,y'=\frac{y}{2}]x>0}{\dfrac{\exists\text{R,cut}\quad x>0 \vdash \exists y\,[x'=-x,y'=\frac{y}{2}]x>0}{\text{DG}\quad x>0 \vdash [x'=-x]x>0}}}$$



Differential Ghosts: Proofs in Extra Dimensions

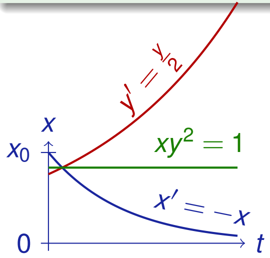
Example (Spooky proof with counterweight ghost)

$$
\dfrac{\dfrac{\dfrac{*}{xy^2{=}1 \vdash x{>}0}\ \mathbb{R} \qquad \dfrac{\dfrac{\dfrac{*}{\vdash -xy^2 + 2xy\frac{y}{2} = 0}\ \mathbb{R}}{\vdash [x'{:=}{-x}][y'{:=}\frac{y}{2}]x'y^2{+}x2yy'{=}0}\ [:=]}{\color{red}{xy^2{=}1} \vdash [x' = -x, y' = \frac{y}{2}]\color{red}{xy^2 = 1}}\ \text{dI}}{\dfrac{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}{\dfrac{x > 0 \vdash \exists y\,[x' = -x, y' = \frac{y}{2}]x > 0}{x > 0 \vdash [x' = -x]x > 0}\ \text{DG}}\ \exists\text{R,cut}}\ \text{MR}
$$



Example (Spooky proof with counterweight ghost)

$$\dfrac{\dfrac{\dfrac{}{*}\;}{xy^2{=}1 \vdash x{>}0}\,\mathbb{R} \qquad \dfrac{\dfrac{\dfrac{*}{\vdash -xy^2 + 2xy\frac{y}{2} = 0}\,\mathbb{R}}{\vdash [x'{:=}{-x}][y'{:=}\frac{y}{2}]x'y^2{+}x2yy'{=}0}\,[:=]}{xy^2{=}1 \vdash [x' = -x, y' = \frac{y}{2}]xy^2 = 1}\,\text{dI}}{\dfrac{xy^2 = 1 \vdash [x' = -x, y' = \frac{y}{2}]x > 0}{\dfrac{x > 0 \vdash \exists y\,[x' = -x, y' = \frac{y}{2}]x > 0}{x > 0 \vdash [x' = -x]x > 0}\,\text{DG}}\,\exists\text{R,cut}}\,\text{MR}$$



Creative proofs with differential ghosts prove what we otherwise couldn't!

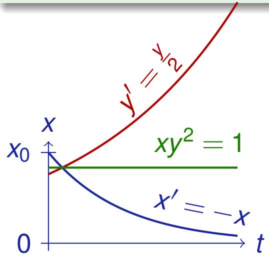
Differential Ghosts: Proofs in Extra Dimensions

Example (Spooky proof with counterweight ghost)

$$\dfrac{\mathbb{R}\;\dfrac{*}{xy^2{=}1 \vdash x{>}0} \qquad \text{dI}\;\dfrac{[:=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash -xy^2+2xy\frac{y}{2}=0}}{\vdash [x'{:=}{-x}][y'{:=}\frac{y}{2}]x'y^2{+}x2yy'{=}0}}{xy^2{=}1 \vdash [x'=-x, y'=\frac{y}{2}]xy^2=1}}{\text{MR}\; xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]x>0}$$

$$\exists\text{R,cut}\quad \dfrac{xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]x>0}{x>0 \vdash \exists y\,[x'=-x, y'=\frac{y}{2}]x>0}$$

$$\text{DG}\quad \dfrac{x>0 \vdash \exists y\,[x'=-x, y'=\frac{y}{2}]x>0}{x>0 \vdash [x'=-x]x>0}$$



Creative proofs with differential ghosts prove what we otherwise couldn't!

Wait, are differential ghosts actually sound?

Constructing Differential Ghosts

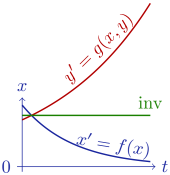
Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x,y) \,\&\, Q]P$$

Constructing Differential Ghosts

Differential Ghost

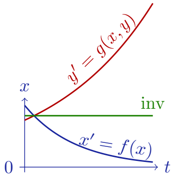
$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x,y) \,\&\, Q]P$$



Constructing Differential Ghosts

Differential Ghost

$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = g(x,y) \,\&\, Q]P$

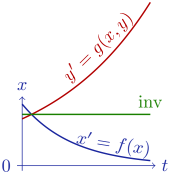


if new $y' = g(x,y)$ has a global solution $y : [0,\infty) \to \mathbb{R}^n$

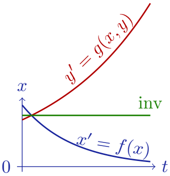
Differential Ghosts

Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$



since new $y' = a(x)y + b(x)$ has a long enough solution



Differential Ghost

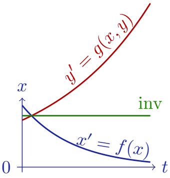
$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$

Differential Ghost

$$\text{dG} \quad \frac{\Gamma \vdash \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

since new $y' = a(x)y + b(x)$ has a long enough solution

Differential Ghosts



Differential Ghost

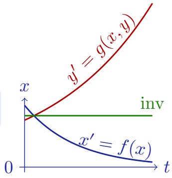
$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$

Differential Ghost

$$\text{dG} \quad \frac{\Gamma \vdash \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

since new $y' = a(x)y + b(x)$ has a long enough solution

Differential Ghosts



Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$

Differential Ghost

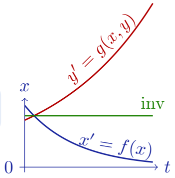
$$\text{dG} \quad \frac{\Gamma \vdash \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

Differential Auxiliary

$$\text{dA} \quad \frac{\vdash F \leftrightarrow \exists y\, G \quad G \vdash [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]G}{F \vdash [x' = f(x) \,\&\, Q]F}$$

since new $y' = a(x)y + b(x)$ has a long enough solution

Differential Ghosts



Differential Ghost

$$[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$$

Differential Ghost

$$\text{dG}\ \frac{\Gamma \vdash \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

Differential Auxiliary

$$\text{dA}\ \frac{\vdash F \leftrightarrow \exists y\, G \quad G \vdash [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]G}{F \vdash [x' = f(x) \,\&\, Q]F}$$

$$\text{DG}\ \dfrac{\text{MR}\ \dfrac{\dfrac{\exists y\, G \vdash F}{G \vdash F} \qquad \exists\text{R,cut}\ \dfrac{F \vdash \exists y\, G \quad G \vdash [x' = f(x), y' = a(x)y + b(x)]G}{F \vdash \exists y\, [x' = f(x), y' = a(x)y + b(x)]G}}{F \vdash \exists y\, [x' = f(x), y' = a(x)y + b(x)]F}}{F \vdash [x' = f(x)]F}$$

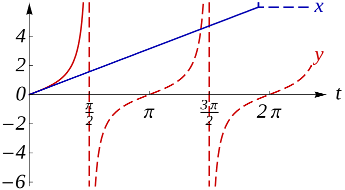
Nonexistent Differential Ghosts

What could possibly go wrong?

$$\dfrac{\dfrac{x{=}0, y{=}0 \vdash [x' = 1, y' = y^2 + 1]\, x \leq 6}{x = 0 \vdash \exists y\, [x' = 1, y' = y^2 + 1]\, x \leq 6}\ \exists R}{x = 0 \vdash [x' = 1]\, x \leq 6}\ \lightning$$

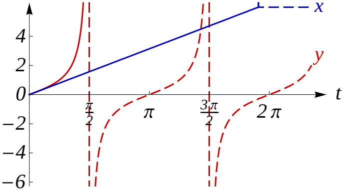
What could possibly go wrong?

$$\dfrac{\dfrac{x{=}0, y{=}0 \vdash [x' = 1, y' = y^2 + 1]\, x \le 6}{x = 0 \vdash \exists y\, [x' = 1, y' = y^2 + 1]\, x \le 6}\ \exists R}{x = 0 \vdash [x' = 1]\, x \le 6}\ \lightning$$



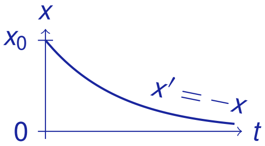
What could possibly go wrong? Explosive ghosts stop the world. Don't!

$$\exists R \frac{x=0, y=0 \vdash [x' = 1, y' = y^2 + 1]\, x \leq 6}{\displaystyle ↯ \frac{x = 0 \vdash \exists y\, [x' = 1, y' = y^2 + 1]\, x \leq 6}{x = 0 \vdash [x' = 1]\, x \leq 6}}$$



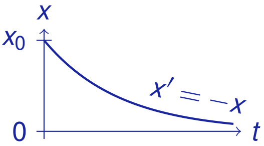
Substitute Ghosts

dA $$x > 0 \vdash [x' = -x]x > 0$$

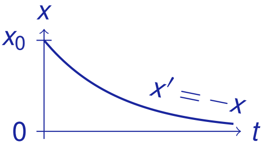


Substitute Ghosts

$$\dfrac{\overset{\mathbb{R}}{\overline{\vdash x>0 \leftrightarrow \exists y\, xy^2=1}} \quad \overset{\text{dI}}{\overline{xy^2=1 \vdash [x' = -x, y' = \boxed{}]xy^2 = 1}}}{x > 0 \vdash [x' = -x]x > 0}\ \text{dA}$$

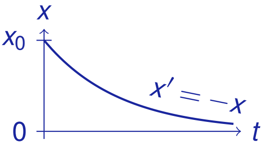


$$
\dfrac{\mathbb{R}\ \dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\ \dfrac{}{xy^2=1 \vdash [x' = -x, y' = \boxed{\quad}]xy^2 = 1}}{x > 0 \vdash [x' = -x]x > 0}\ \text{dA}
$$



Substitute Ghosts

$$
\dfrac{
\mathbb{R}\ \dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1}
\qquad
\text{dI}\ \dfrac{[:=]\ \dfrac{}{\vdash [x':=-x][y':=\square]x'y^2+x2yy'=0}}{xy^2=1 \vdash [x'=-x, y'=\square]xy^2=1}
}{
x>0 \vdash [x'=-x]x>0
}\ \text{dA}
$$

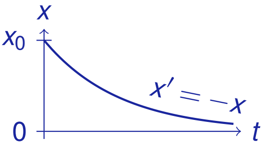


$$\vdash -xy^2 + 2xy\,\square = 0$$

$$[:=]\ \vdash [x' := -x][y' := \square]x'y^2 + x2yy' = 0$$

$$\mathbb{R}\ \dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\ xy^2=1 \vdash [x' = -x, y' = \square]xy^2 = 1$$

$$\text{dA}\ x > 0 \vdash [x' = -x]x > 0$$



$$\text{could prove if } \square = \frac{y}{2}$$

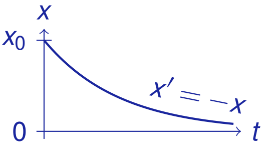
$$\vdash -xy^2 + 2xy\square = 0$$

$$[:=]\quad \vdash [x' := -x][y' := \square]x'y^2 + x2yy' = 0$$

$$*$$

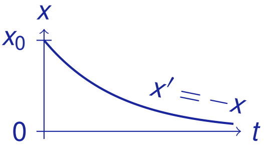
$$\mathbb{R}\ \vdash x>0 \leftrightarrow \exists y\, xy^2=1 \qquad \text{dI}\quad xy^2=1 \vdash [x' = -x, y' = \square]xy^2 = 1$$

$$\text{dA}\quad x > 0 \vdash [x' = -x]x > 0$$



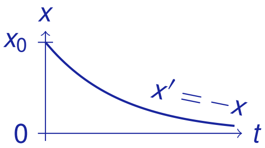
Substitute Ghosts

$$\dfrac{\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1}\;\mathbb{R} \qquad \dfrac{\dfrac{\dfrac{\text{could prove if } ☁ = \frac{y}{2}}{\vdash -xy^2 + 2xy\,☁ = 0}}{\vdash [x':=-x][y':=☁]x'y^2+x2yy'=0}\;[:=]}{xy^2=1 \vdash [x' = -x, y' = ☁]xy^2 = 1}\;\text{dI}}{x > 0 \vdash [x' = -x]x > 0}\;\text{dA}$$



Substitute Ghosts

$$\dfrac{\dfrac{*}{\mathbb{R}\ \vdash x>0 \leftrightarrow \exists y\, xy^2=1} \quad \dfrac{\dfrac{\dfrac{\text{could prove if } \frac{y}{2} = \frac{y}{2} \text{ proved!}}{\vdash -xy^2 + 2xy\frac{y}{2} = 0}}{[:=]\ \vdash [x':=-x][y':=\frac{y}{2}]x'y^2+x2yy'=0}}{\text{dI}\ xy^2=1 \vdash [x'=-x, y'=\frac{y}{2}]xy^2=1}}{\text{dA}\ x>0 \vdash [x'=-x]x>0}$$



Substitute Ghosts

$$\text{could prove if } \textcolor{red}{\frac{y}{2}} = \frac{y}{2}$$

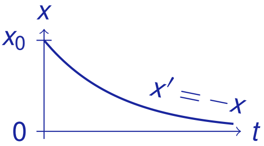
$$\vdash -xy^2 + 2xy\textcolor{red}{\frac{y}{2}} = 0$$

$$[:=]\quad \vdash [x' := -x][y' := \textcolor{red}{\tfrac{y}{2}}]x'y^2 + x2yy' = 0$$

$$*$$

$$\mathbb{R}\quad \vdash x > 0 \leftrightarrow \exists y\, xy^2 = 1 \qquad \text{dI}\quad xy^2 = 1 \vdash [x' = -x, y' = \textcolor{red}{\tfrac{y}{2}}]xy^2 = 1$$

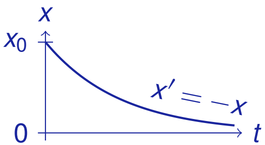
$$\text{dA}\quad x > 0 \vdash [x' = -x]x > 0$$



This is a recipe for brewing suitable differential ghosts!

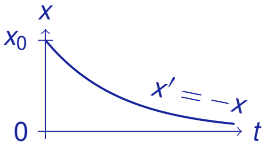
Substitute Ghosts

$$\dfrac{\dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1}\,\mathbb{R} \qquad \dfrac{\dfrac{\dfrac{\text{could prove if } j(y) = \frac{y}{2} \text{ proved!}}{\vdash -xy^2 + 2xy\,j(y) = 0}}{\vdash [x':=-x][y':=j(y)]x'y^2+x2yy'=0}\,[:=]}{xy^2=1 \vdash [x'=-x, y'=j(y)]xy^2=1}\,\text{dI}}{x>0 \vdash [x'=-x]x>0}\,\text{dA}$$



Function symbol $j(y)$ can play the rôle of a substitute ghost

$$\dfrac{\dfrac{*}{\mathbb{R}\ \vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \dfrac{\dfrac{\dfrac{\text{could prove if } \frac{y}{2} \quad = \frac{y}{2}}{\vdash -xy^2 + 2xy\frac{y}{2} \quad = 0}}{[:=]\ \vdash [x':=-x][y':=\frac{y}{2} \quad]x'y^2+x2yy'=0}}{\text{dI}\ xy^2=1 \vdash [x'=-x, y'=\frac{y}{2} \quad]xy^2=1}}{\text{dA}\quad x>0 \vdash [x'=-x]x>0}$$



Function symbol $j(y)$ can be substituted uniformly

Chapter 18

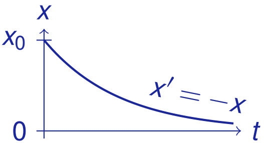
Substitute Ghosts

$$\text{could prove if } j(y) = \frac{y}{2}$$

$$\vdash -xy^2 + 2xy\,j(y) = 0$$

$$[:=]\quad \vdash [x':=-x][y':=j(y)]\,x'y^2+x2yy'=0$$

$$\frac{\mathbb{R}\ \dfrac{*}{\vdash x>0 \leftrightarrow \exists y\, xy^2=1} \qquad \text{dI}\ xy^2=1 \vdash [x'=-x, y'=j(y)]\,xy^2=1}{x>0 \vdash [x'=-x]\,x>0}\ \text{dA}$$



Function symbol $j(y)$ needs to be instantiated linearly in y

1. DG introduces time t, DC cuts solution in, that DI proves and
2. DW exports to postcondition
3. inverse DC removes evolution domain constraints
4. inverse DG removes original ODE
5. DS solves remaining ODE for time $[x'=c()]P \leftrightarrow \forall t{\geq}0\,[x:=x+c()t]P$

$$\begin{array}{ll}
 & * \\
\hline
\mathbb{R} & \Gamma \vdash \forall s{\geq}0\,(x_0 + \frac{a}{2}s^2 + v_0 s \geq 0) \\
\hline
[:=] & \Gamma \vdash \forall s{\geq}0\,[t:=0+1s]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 \\
\hline
\text{DS} & \Gamma \vdash [t'=1]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 \\
\hline
\text{DG} & \Gamma \vdash [v'=a, t'=1]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 \\
\hline
\text{DG} & \Gamma \vdash [x'=v, v'=a, t'=1]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 \quad \triangleright \\
\hline
\text{DC} & \Gamma \vdash [x'=v, v'=a, t'=1 \,\&\, v=v_0+at]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 \quad \triangleright \\
\hline
\text{DC} & \Gamma \vdash [x'=v, v'=a, t'=1 \,\&\, v=v_0+at \land x=x_0+\frac{a}{2}t^2+v_0 t]x_0 + \frac{a}{2}t^2 + v_0 t \geq 0 \\
\hline
\text{MR} & \Gamma \vdash [x'=v, v'=a, t'=1 \,\&\, v=v_0+at \land x=x_0+\frac{a}{2}t^2+v_0 t](x=x_0+\frac{a}{2}t^2+v_0 t \to x{\geq}0) \\
\hline
\text{DW} & \Gamma \vdash [x'=v, v'=a, t'=1 \,\&\, v=v_0+at \land x=x_0+\frac{a}{2}t^2+v_0 t]x \geq 0 \quad \triangleright \\
\hline
\text{DC} & \Gamma \vdash [x'=v, v'=a, t'=1 \,\&\, v=v_0+at]x \geq 0 \quad \triangleright \\
\hline
\text{DC} & \Gamma \vdash [x'=v, v'=a, t'=1]x \geq 0 \quad t:=0 \\
\hline
 & \Gamma \vdash \exists t\,[x'=v, v'=a, t'=1]x \geq 0 \\
\hline
\text{DG} & \Gamma \vdash [x'=v, v'=a]x \geq 0
\end{array}$$

▷ Proving Solutions of Differential Equations

These are the side branches elided above by ▷

$$\text{dI}\ \frac{}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}$$

$$\text{dI}\ \frac{}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}$$

These are the side branches elided above by ▷

$$\text{dI}\ \dfrac{[:=]\ \dfrac{}{\vdash [v' := a][t' := 1]v' = at'}}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}$$

$$\text{dI}\ \dfrac{}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}$$

These are the side branches elided above by ▷

$$\dfrac{\dfrac{\dfrac{}{\vdash a = a \cdot 1}\ \mathbb{R}}{\vdash [v' := a][t' := 1]v' = at'}\ [:=]}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}\ \text{dI}$$

$$\dfrac{}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}\ \text{dI}$$

These are the side branches elided above by ▷

$$\text{dI}\ \dfrac{[:=]\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash a = a \cdot 1}}{\vdash [v' := a][t' := 1]v' = at'}}{\phi \vdash [x' = v, v' = a, t' = 1]\color{red}{v = v_0 + at}}$$

$$\text{dI}\ \dfrac{}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, \color{red}{v = v_0 + at}]x = x_0 + \frac{a}{2}t^2 + v_0 t}$$

These are the side branches elided above by ▷

$$\dfrac{\dfrac{\dfrac{*}{\vdash a = a\cdot 1}\ \mathbb{R}}{\vdash [v' := a][t' := 1]v' = at'}\ [:=]}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}\ \text{dI}$$

$$\dfrac{\dfrac{}{\vdash v = v_0 + at \rightarrow [x' := v][t' := 1]x' = at t' + v_0 t'}\ [:=]}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}\ \text{dI}$$

These are the side branches elided above by $\triangleright$

$$\dfrac{\dfrac{\dfrac{*}{\vdash a = a \cdot 1}\ \mathbb{R}}{\vdash [v' := a][t' := 1]v' = at'}\ [:=]}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}\ \text{dI}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash v = v_0 + at \to v = at \cdot 1 + v_0 \cdot 1}\ \mathbb{R}}{\vdash v = v_0 + at \to [x' := v][t' := 1]x' = att' + v_0 t'}\ [:=]}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}\ \text{dI}$$

These are the side branches elided above by ▷

$$\dfrac{\mathbb{R}\dfrac{*}{\vdash a = a\cdot 1}}{[:=]\dfrac{\vdash [v' := a][t' := 1]v' = at'}{\text{dI}\ \phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}}$$

$$\dfrac{\mathbb{R}\dfrac{*}{\vdash v = v_0 + at \rightarrow v = at\cdot 1 + v_0\cdot 1}}{[:=]\dfrac{\vdash v = v_0 + at \rightarrow [x' := v][t' := 1]x' = att' + v_0 t'}{\text{dI}\ \phi \vdash [x' = v, v' = a, t' = 1\ \&\ v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}}$$

These are the side branches elided above by ▷

$$\dfrac{\dfrac{\dfrac{*}{\vdash a = a \cdot 1}\ \mathbb{R}}{\vdash [v' := a][t' := 1]v' = at'}\ [:=]}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}\ \text{dI}$$

$$\dfrac{\dfrac{\dfrac{*}{\vdash v = v_0 + at \to v = at \cdot 1 + v_0 \cdot 1}\ \mathbb{R}}{\vdash v = v_0 + at \to [x' := v][t' := 1]x' = att' + v_0 t'}\ [:=]}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}\ \text{dI}$$

But ϕ needs $v = v_0 \land x = x_0$ initially for dI

These are the side branches elided above by ▷

$$\dfrac{\dfrac{\dfrac{*}{\vdash a = a\cdot 1}\ \mathbb{R}}{\vdash [v' := a][t' := 1]v' = at'}\ [:=]}{\phi \vdash [x' = v, v' = a, t' = 1]v = v_0 + at}\ \mathrm{dI}$$

$$\dfrac{\dfrac{\dfrac{*}{\vdash v = v_0 + at \rightarrow v = at\cdot 1 + v_0\cdot 1}\ \mathbb{R}}{\vdash v = v_0 + at \rightarrow [x' := v][t' := 1]x' = att' + v_0 t'}\ [:=]}{\phi \vdash [x' = v, v' = a, t' = 1 \,\&\, v = v_0 + at]x = x_0 + \frac{a}{2}t^2 + v_0 t}\ \mathrm{dI}$$

But ϕ needs $v = v_0 \land x = x_0$ initially for dI

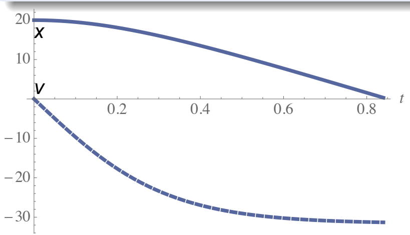
Discrete ghosts to the rescue: $[x_0 := x][v_0 := v]\ldots$

who can remember initial value on demand.

Limit Velocity of an Aerodynamic Ball

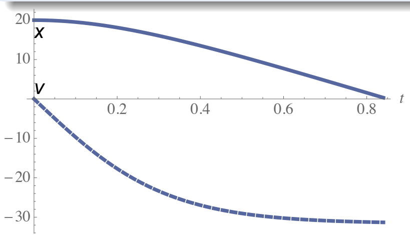
Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \qquad \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]$$



Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \quad \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]$$

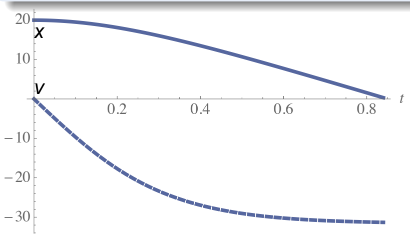


Equilibrium

$v' = 0$ iff $-g + rv^2 = 0$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \quad \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]$$



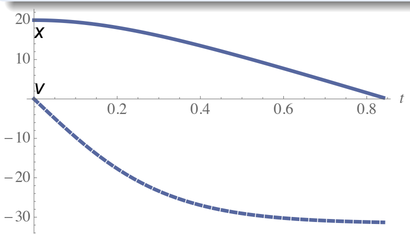
Equilibrium

$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

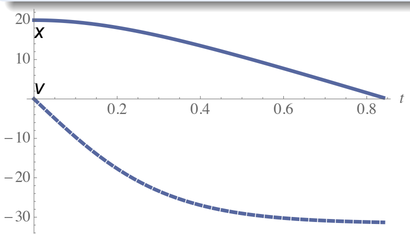
$v'=0$ iff $-g+rv^2=0$ iff $v=\pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\text{dA}\ \frac{}{v>-\sqrt{g/r} \vdash [x' = v, v' = -g + rv^2]\, v > -\sqrt{g/r}}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

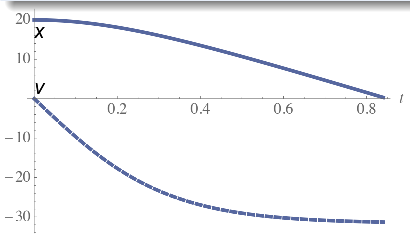
Limit Velocity of an Aerodynamic Ball

$$\text{dI}\ \frac{y^2(v+\sqrt{g/r})=1 \vdash [x'=v, v'=-g+rv^2, {\color{red}y'=j(x,v,y)}]\, y^2(v+\sqrt{g/r})=1 \quad \triangleright}{}$$

$$\text{dA}\ \frac{}{v>-\sqrt{g/r} \vdash [x'=v, v'=-g+rv^2]\, v > -\sqrt{g/r}}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

$v' = 0$ iff $-g+rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

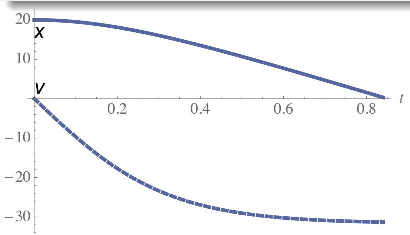
Limit Velocity of an Aerodynamic Ball

$$\dfrac{\dfrac{\dfrac{\vdash [x':=v][v':=-g+rv^2][y':=j(x,v,y)]2yy'(v+\sqrt{g/r})+y^2v'=0}{y^2(v+\sqrt{g/r})=1 \vdash [x'=v, v'=-g+rv^2, y'=j(x,v,y)]y^2(v+\sqrt{g/r})=1}\text{dI}}{v>-\sqrt{g/r} \vdash [x'=v, v'=-g+rv^2]\, v>-\sqrt{g/r}}\text{dA}}{}[:=]$$

▷

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\, v>-\sqrt{\frac{g}{r}}$$



Equilibrium

$v'=0$ iff $-g+rv^2=0$ iff $v=\pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$\vdash 2y(j(x,v,y))(v+\sqrt{g/r}) + y^2(-g+rv^2) = 0$$

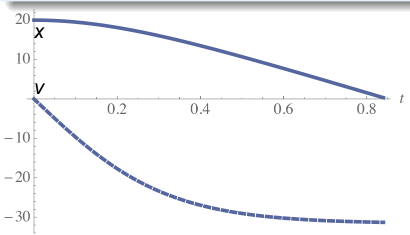
[:=]
$$\vdash [x':=v][v':=-g+rv^2][y':=j(x,v,y)]\,2yy'(v+\sqrt{g/r})+y^2v'=0$$

dI
$$y^2(v+\sqrt{g/r})=1 \vdash [x'=v, v'=-g+rv^2, y'=j(x,v,y)]\,y^2(v+\sqrt{g/r})=1 \quad \triangleright$$

dA
$$v>-\sqrt{g/r} \vdash [x'=v, v'=-g+rv^2]\,v>-\sqrt{g/r}$$

Proposition (Aerodynamic velocity limits)

$$g>0 \land r>0 \land v>-\sqrt{\frac{g}{r}} \rightarrow [x'=v, v'=-g+rv^2 \,\&\, x\geq 0 \land v\leq 0]\,v>-\sqrt{\frac{g}{r}}$$



Equilibrium

$v'=0$ iff $-g+rv^2=0$ iff $v=\pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$\mathbb{R}$
$$\vdash -ry^2(v^2 - g/r) + y^2(-g + rv^2) = 0$$

$$\vdash 2y(-r/2(v - \sqrt{g/r})y)(v + \sqrt{g/r}) + y^2(-g + rv^2) = 0$$

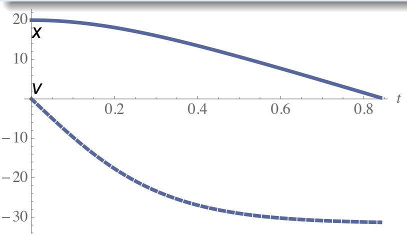
[:=]
$$\vdash [x' := v][v' := -g + rv^2][y' := -r/2(v - \sqrt{g/r})y]2yy'(v + \sqrt{g/r}) + y^2 v' = 0$$

dI
$$y^2(v + \sqrt{g/r}) = 1 \vdash [x' = v, v' = -g + rv^2, y' = -r/2(v - \sqrt{g/r})y]y^2(v + \sqrt{g/r}) = 1 \quad \triangleright$$

dA
$$v > -\sqrt{g/r} \vdash [x' = v, v' = -g + rv^2]\, v > -\sqrt{g/r}$$

Proposition (Aerodynamic velocity limits)

$$g > 0 \land r > 0 \land v > -\sqrt{\frac{g}{r}} \to [x' = v, v' = -g + rv^2 \,\&\, x \geq 0 \land v \leq 0]\, v > -\sqrt{\frac{g}{r}}$$



Equilibrium

$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

Limit Velocity of an Aerodynamic Ball

$$*$$

$\mathbb{R}$
$$\vdash -ry^2(v^2 - g/r) + y^2(-g + rv^2) = 0$$

$$\vdash 2y\left(-\tfrac{r}{2}\left(v - \sqrt{g/r}\right)y\right)\left(v + \sqrt{g/r}\right) + y^2(-g + rv^2) = 0$$

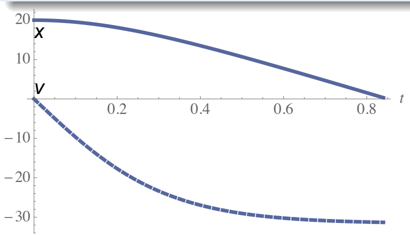
[:=]
$$\vdash [x' := v][v' := -g + rv^2][y' := -\tfrac{r}{2}(v - \sqrt{g/r})y]\, 2yy'(v + \sqrt{g/r}) + y^2 v' = 0$$

dI
$$y^2(v + \sqrt{g/r}) = 1 \vdash [x' = v, v' = -g + rv^2, y' = -\tfrac{r}{2}(v - \sqrt{g/r})y]\, y^2(v + \sqrt{g/r}) = 1 \quad \triangleright$$

dA
$$v > -\sqrt{g/r} \vdash [x' = v, v' = -g + rv^2]\, v > -\sqrt{g/r}$$

Proposition (Aerodynamic velocity limits)

$$g > 0 \land r > 0 \land v > -\sqrt{\frac{g}{r}} \rightarrow [x' = v, v' = -g + rv^2 \,\&\, x \geq 0 \land v \leq 0]\, v > -\sqrt{\frac{g}{r}}$$



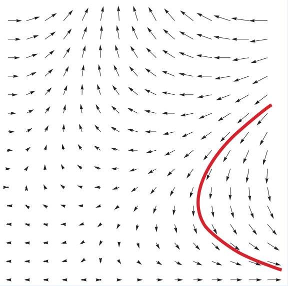
Equilibrium

$v' = 0$ iff $-g + rv^2 = 0$ iff $v = \pm\sqrt{\frac{g}{r}}$

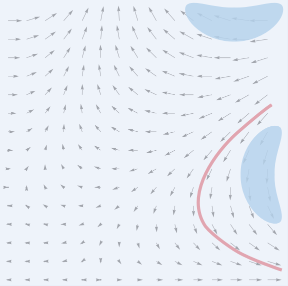
Outline

Differential Invariants for Differential Equations

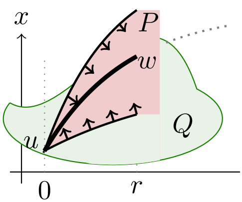
Differential Invariant



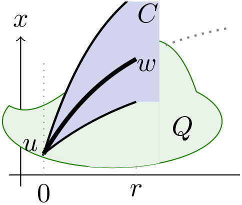
Differential Cut



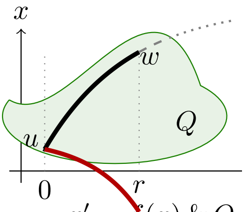
Differential Ghost



$x' = f(x) \,\&\, Q$



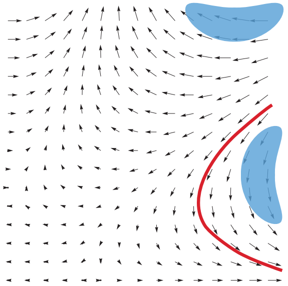
$x' = f(x) \,\&\, Q$



$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

Differential Invariant



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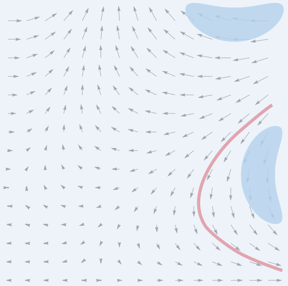
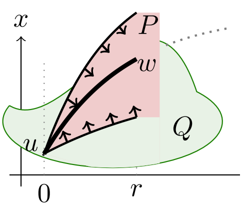
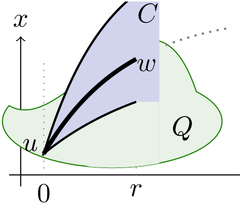


Differential Ghost



$x' = f(x) \,\&\, Q$



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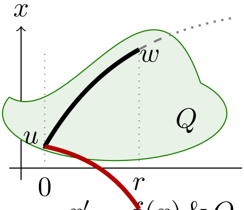
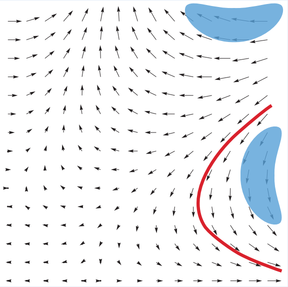
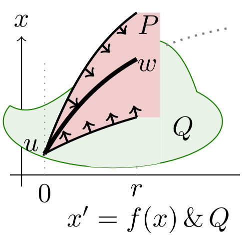


$x' = f(x) \,\&\, Q$

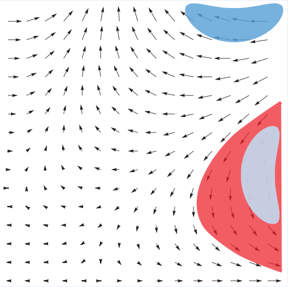
Differential Invariants for Differential Equations

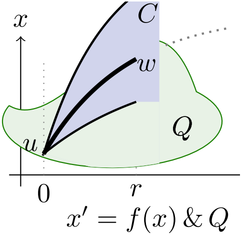
Differential Invariant



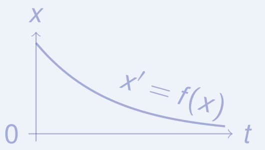


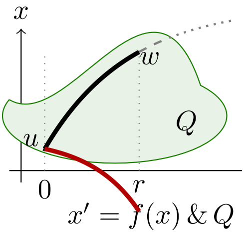
Differential Cut





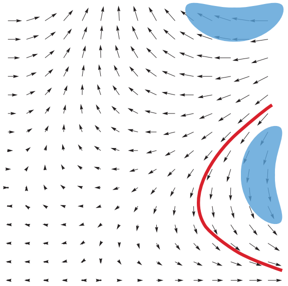
Differential Ghost

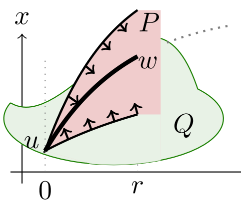




Differential Invariants for Differential Equations

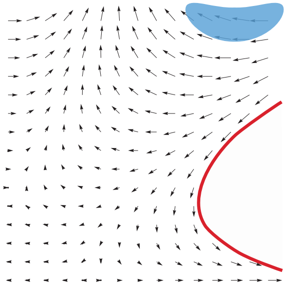
Differential Invariant

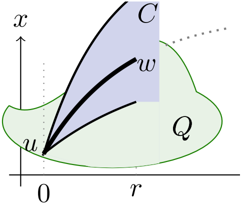




$x' = f(x) \,\&\, Q$

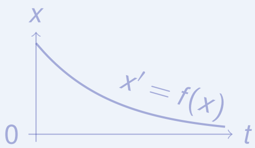
Differential Cut

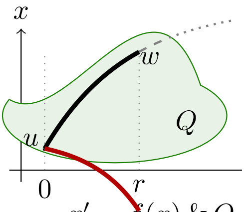




$x' = f(x) \,\&\, Q$

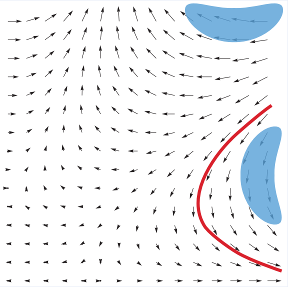
Differential Ghost



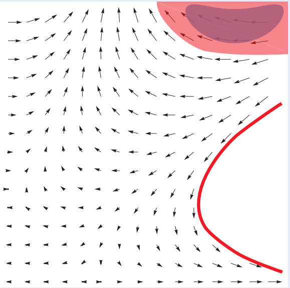


$x' = f(x) \,\&\, Q$

Differential Invariant



Differential Cut



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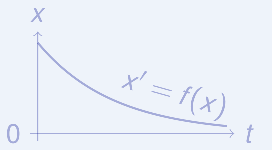
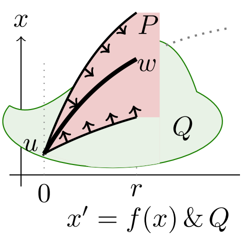
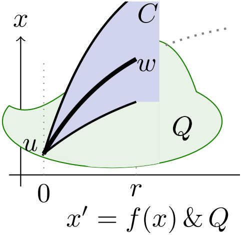
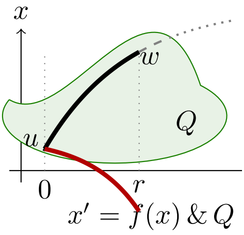


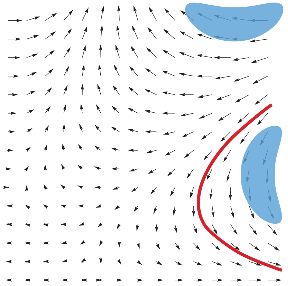




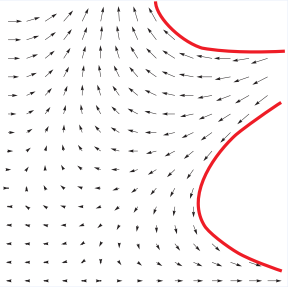


Differential Invariants for Differential Equations

Differential Invariant



Differential Cut



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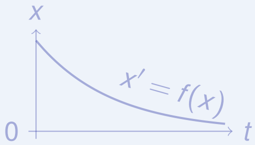
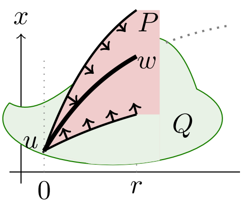
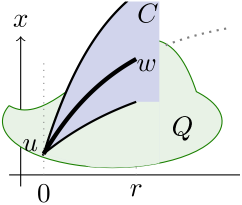
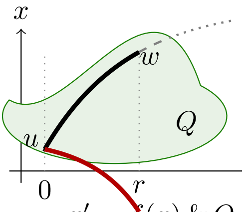




$x' = f(x) \,\&\, Q$



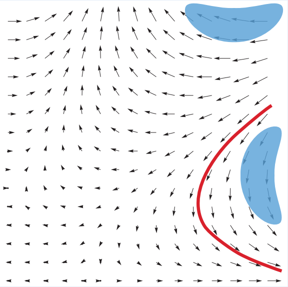
$x' = f(x) \,\&\, Q$



$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

Differential Invariant



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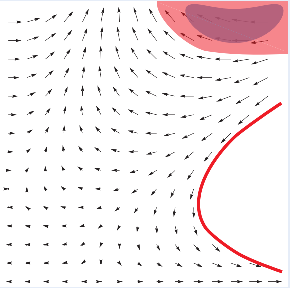

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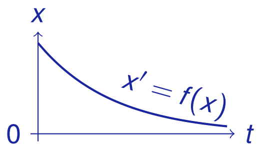
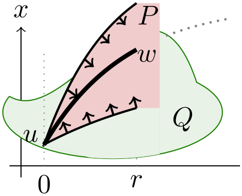
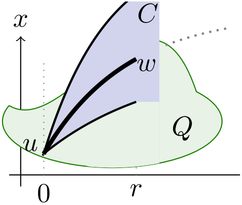
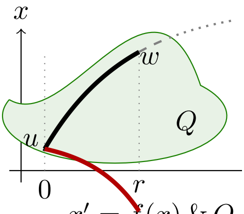
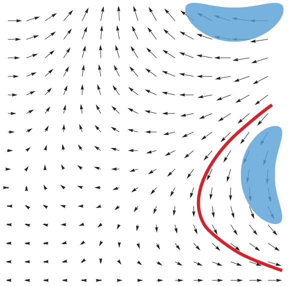




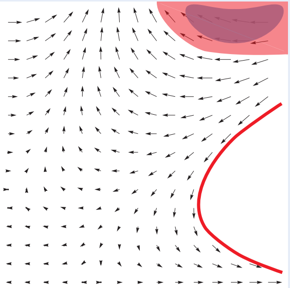




Differential Invariant



Differential Cut



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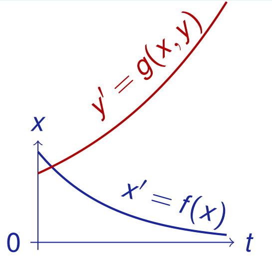
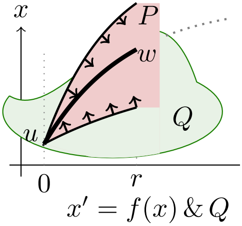
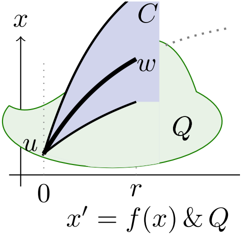
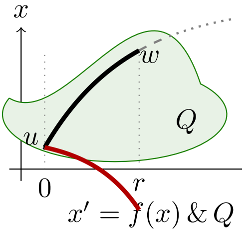


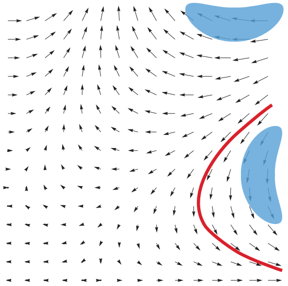


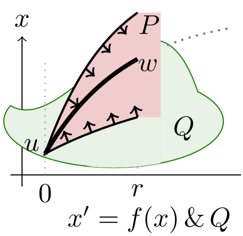




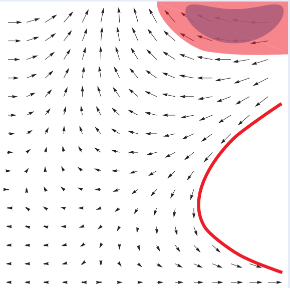
Differential Invariants for Differential Equations

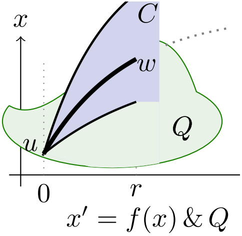
Differential Invariant



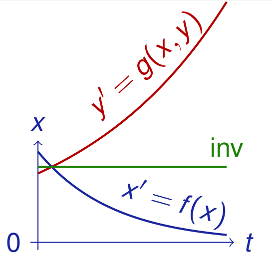


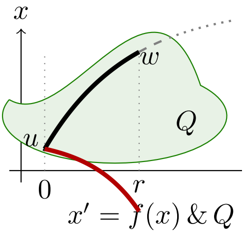
Differential Cut





Differential Ghost





Differential Invariant

$$\frac{Q \vdash [x' := f(x)](P)'}{P \vdash [x' = f(x) \,\&\, Q]P}$$

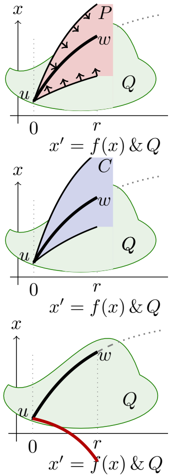
Differential Cut

$$\frac{P \vdash [x' = f(x) \,\&\, Q]C \quad P \vdash [x' = f(x) \,\&\, Q \wedge C]P}{P \vdash [x' = f(x) \,\&\, Q]P}$$

Differential Ghost

$$\frac{P \leftrightarrow \exists y\, G \quad G \vdash [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \vdash [x' = f(x) \,\&\, Q]P}$$

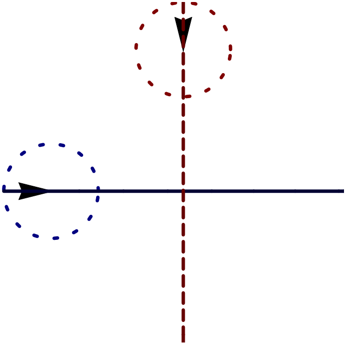
if new $y' = g(x, y)$ has long enough solution

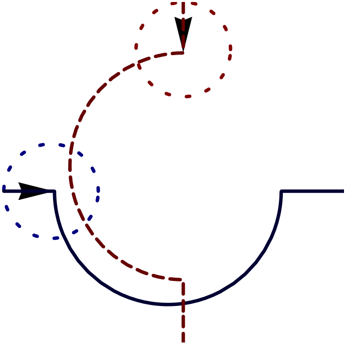


Outline

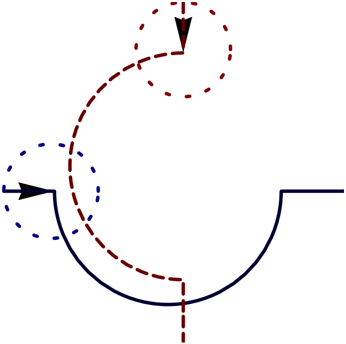
6. Appendix
   - Axiomatic Ghosts
   - Arithmetic Ghosts
   - Nondeterministic Assignments & Ghosts of Choice
   - Differential-Algebraic Ghosts

Air Traffic Control



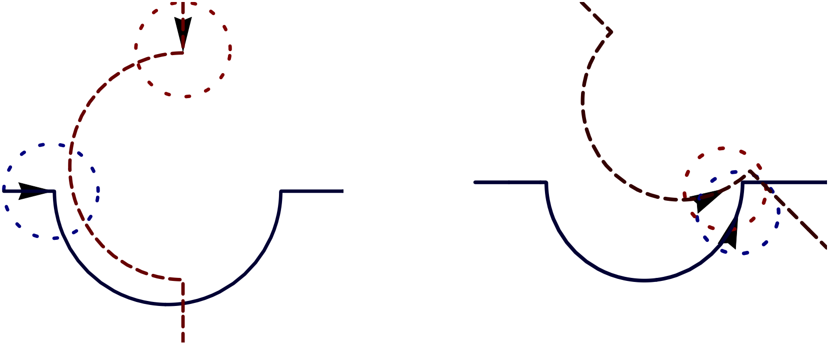


Air Traffic Control



Verification?

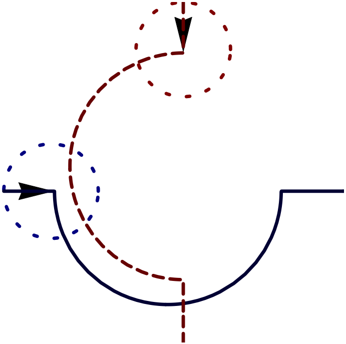
looks correct

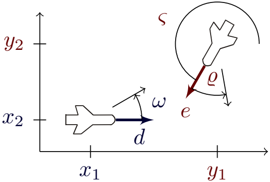


Verification?

looks correct NO!

Air Traffic Control

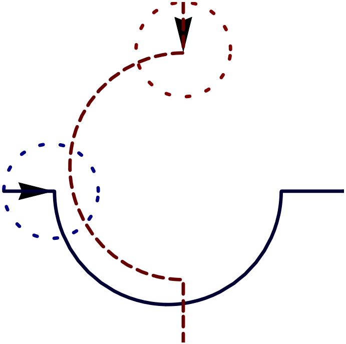


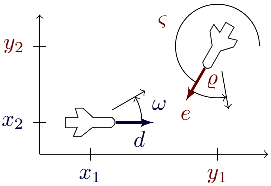


$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Verification?

looks correct NO!

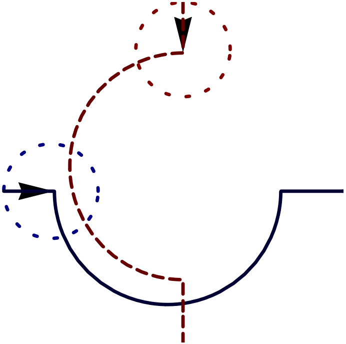


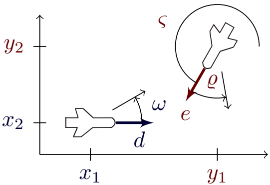


$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Example ("Solving" differential equations)

$$x_1(t) = \frac{1}{\omega\varpi}\big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$
$$+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$
$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\big)\ldots$$





$$\begin{bmatrix} x_1' = -v_1 + v_2 \cos\vartheta + \omega x_2 \\ x_2' = v_2 \sin\vartheta - \omega x_1 \\ \vartheta' = \varpi - \omega \end{bmatrix}$$

Example ("Solving" differential equations)

$$\forall t \geq 0 \quad \frac{1}{\omega\varpi}\Big(x_1\omega\varpi\cos t\omega - v_2\omega\cos t\omega\sin\vartheta + v_2\omega\cos t\omega\cos t\varpi\sin\vartheta - v_1\varpi\sin t\omega$$
$$+ x_2\omega\varpi\sin t\omega - v_2\omega\cos\vartheta\cos t\varpi\sin t\omega - v_2\omega\sqrt{1-\sin\vartheta^2}\sin t\omega$$
$$+ v_2\omega\cos\vartheta\cos t\omega\sin t\varpi + v_2\omega\sin\vartheta\sin t\omega\sin t\varpi\Big)\ldots$$

```
\forall R ts2.
  (   0 <= ts2 & ts2 <= t2_0
   ->       (  (om_1)^-1
             * (omb_1)^-1
             * (  om_1 * omb_1 * x1 * cos(om_1 * ts2)
                + om_1 * v2 * cos(om_1 * ts2) * (1 + -1 * (cos(u))^2)^(1 / 2)
                + -1 * omb_1 * v1 * sin(om_1 * ts2)
                + om_1 * omb_1 * x2 * sin(om_1 * ts2)
                + om_1 * v2 * cos(u) * sin(om_1 * ts2)
                + -1 * om_1 * v2 * cos(omb_1 * ts2) * cos(u) * sin(om_1 * ts2)
                + om_1 * v2 * cos(om_1 * ts2) * cos(u) * sin(omb_1 * ts2)
                + om_1 * v2 * cos(om_1 * ts2) * cos(omb_1 * ts2) * sin(u)
                + om_1 * v2 * sin(om_1 * ts2) * sin(omb_1 * ts2) * sin(u)))
        ^2
       +    (  (om_1)^-1
             * (omb_1)^-1
             * (  -1 * omb_1 * v1 * cos(om_1 * ts2)
                + om_1 * omb_1 * x2 * cos(om_1 * ts2)
                + omb_1 * v1 * (cos(om_1 * ts2))^2
                + om_1 * v2 * cos(om_1 * ts2) * cos(u)
                + -1 * om_1 * v2 * cos(om_1 * ts2) * cos(omb_1 * ts2) * cos(u)
                + -1 * om_1 * omb_1 * x1 * sin(om_1 * ts2)
                +   -1
                  * om_1
                  * v2
                  * (1 + -1 * (cos(u))^2)^(1 / 2)
                  * sin(om_1 * ts2)
                + omb_1 * v1 * (sin(om_1 * ts2))^2
                + -1 * om_1 * v2 * cos(u) * sin(om_1 * ts2) * sin(omb_1 * ts2)
                + -1 * om_1 * v2 * cos(omb_1 * ts2) * sin(om_1 * ts2) * sin(u)
                + om_1 * v2 * cos(om_1 * ts2) * sin(omb_1 * ts2) * sin(u)))
        ^2
      >= (p)^2),
 t2_0 >= 0,
 x1^2 + x2^2 >= (p)^2
==>
```

```
\forall R t7.
  (   t7 >= 0
   ->       (  (om_3)^-1
             * (    om_3
                  * (  (om_1)^-1
                     * (omb_1)^-1
                     * (  om_1 * omb_1 * x1 * cos(om_1 * t2_0)
                        +   om_1
                          * v2
                          * cos(om_1 * t2_0)
                          * (1 + -1 * (cos(u))^2)^(1 / 2)
                        + -1 * omb_1 * v1 * sin(om_1 * t2_0)
                        + om_1 * omb_1 * x2 * sin(om_1 * t2_0)
                        + om_1 * v2 * cos(u) * sin(om_1 * t2_0)
                        +   -1
                          * om_1
                          * v2
                          * cos(omb_1 * t2_0)
                          * cos(u)
                          * sin(om_1 * t2_0)
                        +   om_1
                          * v2
                          * cos(om_1 * t2_0)
                          * cos(u)
                          * sin(omb_1 * t2_0)
                        +   om_1
                          * v2
                          * cos(om_1 * t2_0)
                          * cos(omb_1 * t2_0)
                          * sin(u)
                        +   om_1
                          * v2
                          * sin(om_1 * t2_0)
                          * sin(omb_1 * t2_0)
                          * sin(u)))
```

```
    * cos(om_3 * t5)
  +   v2
    * cos(om_3 * t5)
    *    (  1
           +   -1
             * (cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4))^2)
     ^(1 / 2)
  + -1 * v1 * sin(om_3 * t5)
  +   om_3
    * (   (om_1)^-1
       * (omb_1)^-1
       * (   -1 * omb_1 * v1 * cos(om_1 * t2_0)
          + om_1 * omb_1 * x2 * cos(om_1 * t2_0)
          + omb_1 * v1 * (cos(om_1 * t2_0))^2
          + om_1 * v2 * cos(om_1 * t2_0) * cos(u)
          +   -1
            * om_1
            * v2
            * cos(om_1 * t2_0)
            * cos(omb_1 * t2_0)
            * cos(u)
          + -1 * om_1 * omb_1 * x1 * sin(om_1 * t2_0)
          +   -1
            * om_1
            * v2
            * (1 + -1 * (cos(u))^2)^(1 / 2)
            * sin(om_1 * t2_0)
          + omb_1 * v1 * (sin(om_1 * t2_0))^2
          +   -1
            * om_1
            * v2
            * cos(u)
            * sin(om_1 * t2_0)
            * sin(omb_1 * t2_0)
```

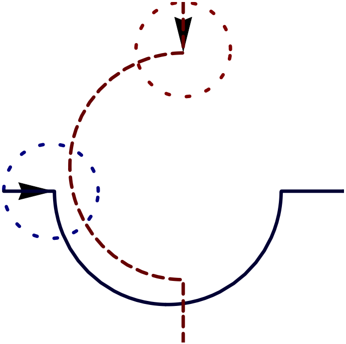
```
                  +   -1
                    * om_1
                    * v2
                    * cos(omb_1 * t2_0)
                    * sin(om_1 * t2_0)
                    * sin(u)
                  +   om_1
                    * v2
                    * cos(om_1 * t2_0)
                    * sin(omb_1 * t2_0)
                    * sin(u)))
          * sin(om_3 * t5)
        +   v2
          * cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
          * sin(om_3 * t5)
        +   v2
          * (cos(om_3 * t5))^2
          * sin(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
        +   v2
          * (sin(om_3 * t5))^2
          * sin(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)))
 ^2
+   (  (om_3)^-1
     * (  -1 * v1 * cos(om_3 * t5)
        +   om_3
          * (  (om_1)^-1
             * (omb_1)^-1
             * (  -1 * omb_1 * v1 * cos(om_1 * t2_0)
                + om_1 * omb_1 * x2 * cos(om_1 * t2_0)
                + omb_1 * v1 * (cos(om_1 * t2_0))^2
                + om_1 * v2 * cos(om_1 * t2_0) * cos(u)
                +   -1
                  * om_1
                  * v2
                  * cos(om_1 * t2_0)
                  * cos(omb_1 * t2_0)
```

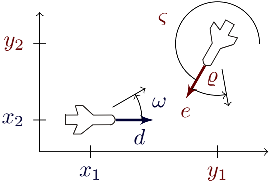
```
            + -1 * om_1 * omb_1 * x1 * sin(om_1 * t2_0)
            +   -1
              * om_1
              * v2
              * (1 + -1 * (cos(u))^2)^(1 / 2)
              * sin(om_1 * t2_0)
            + omb_1 * v1 * (sin(om_1 * t2_0))^2
            +   -1
              * om_1
              * v2
              * cos(u)
              * sin(om_1 * t2_0)
              * sin(omb_1 * t2_0)
            +   -1
              * om_1
              * v2
              * cos(omb_1 * t2_0)
              * sin(om_1 * t2_0)
              * sin(u)
            +   om_1
              * v2
              * cos(om_1 * t2_0)
              * sin(omb_1 * t2_0)
              * sin(u)))
  * cos(om_3 * t5)
+ v1 * (cos(om_3 * t5))^2
+   v2
  * cos(om_3 * t5)
  * cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
+   -1
  * v2
  * (cos(om_3 * t5))^2
  * cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
```

```
+   -1
  * om_3
  * (  (om_1)^-1
     * (omb_1)^-1
     * (  om_1 * omb_1 * x1 * cos(om_1 * t2_0)
        +   om_1
          * v2
          * cos(om_1 * t2_0)
          * (1 + -1 * (cos(u))^2)^(1 / 2)
        + -1 * omb_1 * v1 * sin(om_1 * t2_0)
        + om_1 * omb_1 * x2 * sin(om_1 * t2_0)
        + om_1 * v2 * cos(u) * sin(om_1 * t2_0)
        +   -1
          * om_1
          * v2
          * cos(omb_1 * t2_0)
          * cos(u)
          * sin(om_1 * t2_0)
        +   om_1
          * v2
          * cos(om_1 * t2_0)
          * cos(u)
          * sin(omb_1 * t2_0)
        +   om_1
          * v2
          * cos(om_1 * t2_0)
          * cos(omb_1 * t2_0)
          * sin(u)
        +   om_1
          * v2
          * sin(om_1 * t2_0)
          * sin(omb_1 * t2_0)
          * sin(u)))
  * sin(om_3 * t5)
```

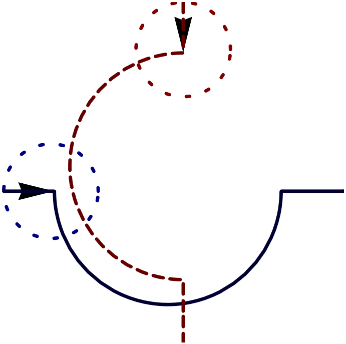
```
        +   -1
          * v2
          *   (  1
                +   -1
                  * (cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4))^2)
           ^(1 / 2)
          * sin(om_3 * t5)
        + v1 * (sin(om_3 * t5))^2
        +   -1
          * v2
          * cos(-1 * om_1 * t2_0 + omb_1 * t2_0 + u + Pi / 4)
          * (sin(om_3 * t5))^2))
  ^2
>= (p)^2)
```

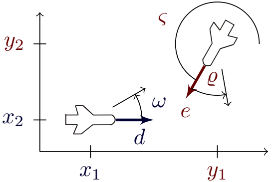
This is just one branch to prove for aircraft ...



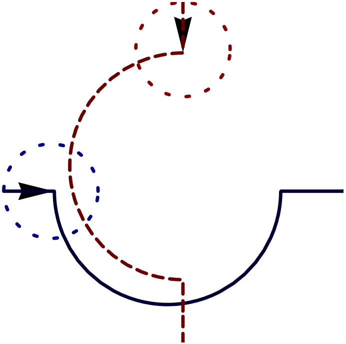


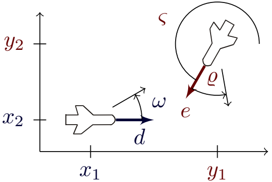
$$\begin{bmatrix} x_1' = v\cos\vartheta & y_1' = u\cos\varsigma \\ x_2' = v\sin\vartheta & y_2' = u\sin\varsigma \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$



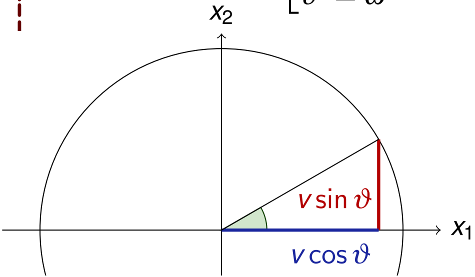


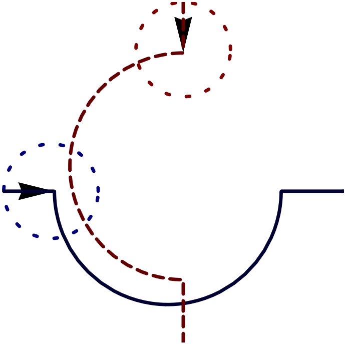
$$\begin{bmatrix} x_1' = v\cos\vartheta & y_1' = u\cos\varsigma \\ x_2' = v\sin\vartheta & y_2' = u\sin\varsigma \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

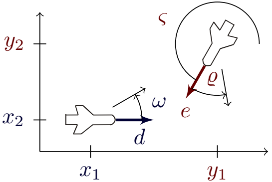




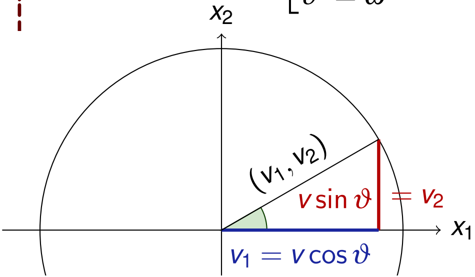
$$\begin{bmatrix} x_1' = v\cos\vartheta & y_1' = u\cos\varsigma \\ x_2' = v\sin\vartheta & y_2' = u\sin\varsigma \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

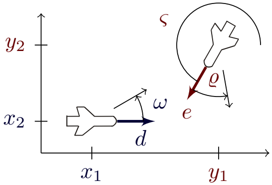






$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

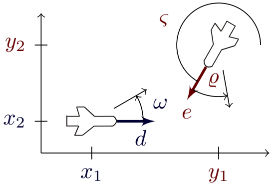




$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = & u_1' = \\ v_2' = & u_2' = \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

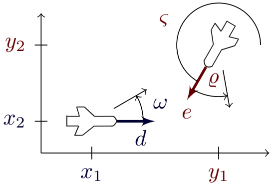
$$v_1' =$$
$$v_2' =$$

Differential Axiomatization of Flight Dynamics



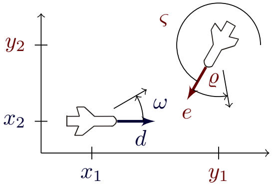
$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = & u_1' = \\ v_2' = & u_2' = \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)'$$
$$v_2' = (v\sin\vartheta)'$$



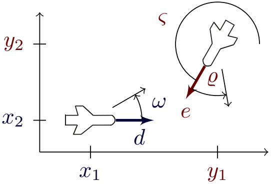
$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = & u_1' = \\ v_2' = & u_2' = \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta'$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta'$$



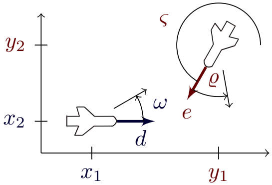
$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = & u_1' = \\ v_2' = & u_2' = \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega$$



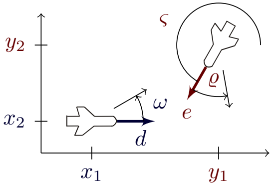
$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = -\omega v_2 & u_1' = \\ v_2' = \omega v_1 & u_2' = \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega = -\omega v_2$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega = \omega v_1$$



$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = -\omega v_2 & u_1' = -\rho u_2 \\ v_2' = \omega v_1 & u_2' = \rho u_1 \\ \vartheta' = \omega & \varsigma' = \rho \end{bmatrix}$$

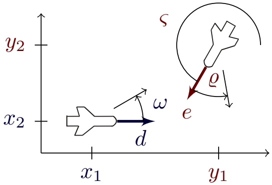
$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega = -\omega v_2$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega = \omega v_1$$



$$\begin{bmatrix} x_1' = v\cos\vartheta = v_1 & y_1' = u\cos\varsigma = u_1 \\ x_2' = v\sin\vartheta = v_2 & y_2' = u\sin\varsigma = u_2 \\ v_1' = -\omega v_2 & u_1' = -\rho u_2 \\ v_2' = \omega v_1 & u_2' = \rho u_1 \\ & \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega = -\omega v_2$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega = \omega v_1$$

$$v = \|(v_1, v_2)\| = \sqrt{v_1^2 + v_2^2}$$



$$\begin{bmatrix} x_1' = v_1 & y_1' = u_1 \\ x_2' = v_2 & y_2' = u_2 \\ v_1' = -\omega v_2 & u_1' = -\rho u_2 \\ v_2' = \omega v_1 & u_2' = \rho u_1 \\ & \end{bmatrix}$$

$$v_1' = (v\cos\vartheta)' = v'\cos\vartheta + v(-\sin\vartheta)\vartheta' = -(v\sin\vartheta)\omega = -\omega v_2$$
$$v_2' = (v\sin\vartheta)' = v'\sin\vartheta + v(\cos\vartheta)\vartheta' = (v\cos\vartheta)\omega = \omega v_1$$

$$v = \|(v_1, v_2)\| = \sqrt{v_1^2 + v_2^2}$$

Arithmetic Ghosts

Syntax $e ::= x \mid x' \mid f(e) \mid e+k \mid e \cdot k \mid (e)'$

Syntax $\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$

Syntax $P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

Wait, what about . . .

Arithmetic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e + k \mid e \cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k$

Arithmetic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \stackrel{\text{def}}{=} e + (-1)\cdot k$

Arithmetic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e \cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \stackrel{\text{def}}{=} e + (-1) \cdot k$
2. $-e$

Arithmetic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \stackrel{\text{def}}{=} e + (-1)\cdot k$
2. $-e \stackrel{\text{def}}{=} 0 - e$

Arithmetic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e \cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \stackrel{\text{def}}{=} e + (-1) \cdot k$
2. $-e \stackrel{\text{def}}{=} 0 - e$
3. e^n

Arithmetic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e \cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \stackrel{\text{def}}{=} e + (-1) \cdot k$
2. $-e \stackrel{\text{def}}{=} 0 - e$
3. $e^n \stackrel{\text{def}}{=} e \cdot \ldots \cdot e$

Arithmetic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e \cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

1. $e - k \stackrel{\text{def}}{=} e + (-1) \cdot k$
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4. e/k

Arithmetic Ghosts

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Arithmetic Ghosts

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4. $e/k \overset{\text{def}}{=}$ depends $\quad q = \frac{b}{c} \overset{\text{def}}{\equiv} qc = b$

   $$q := \frac{b}{c}$$

Arithmetic Ghosts / Ghosts of Division

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
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$$q := \frac{b}{c} \quad \leadsto \quad q := *;\ ?qc = b$$

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$$q := \frac{b}{c} \quad \rightsquigarrow \quad q := *;\ ?qc = b \wedge c \neq 0$$

$$x := 2 + \frac{b}{c} + e$$

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
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$$q := \frac{b}{c} \quad\rightsquigarrow\quad q:=*;\ ?qc=b\wedge c\neq 0$$

$$x := 2+\frac{b}{c}+e \quad\rightsquigarrow\quad q:=*;\ ?qc=b\wedge c\neq 0;\ x:=2+q+e$$

Arithmetic Ghosts / Ghosts of Division

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
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$$q := \frac{b}{c} \quad \rightsquigarrow \quad q := *;\ ?qc = b \land c \neq 0$$

$$x := 2 + \frac{b}{c} + e \quad \rightsquigarrow \quad q := *;\ ?qc = b \land c \neq 0;\ x := 2 + q + e$$

$$x := a + \sqrt{4y}$$

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
Syntax	$\alpha ::= x:=e \mid ?Q \mid x'=f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$
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$$q:=\frac{b}{c} \quad\rightsquigarrow\quad q:=*;\ ?qc=b\wedge c\neq 0$$

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$$x:=a+\sqrt{4y} \quad\rightsquigarrow\quad q:=*;\ ?q^2=4y;\ x:=a+q$$

Arithmetic Ghosts / Ghosts of Division

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Arithmetic Ghosts / Ghosts of Division

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4. **arithmetic ghost: auxiliary for the model** ere $c \neq 0$

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nondeterministic assignment $q := *$ not in syntax

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Nondeterministic Assignments & Ghosts of Choice

Nondeterministic assignment $x := *$ not in HP syntax.

Nondeterministic Assignments & Ghosts of Choice

Nondeterministic assignment $x := *$ not in HP syntax. ① Modular add

Syntax $\alpha ::= \ldots \mid x := *$

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Semantics $[\![x := *]\!] = \{(\omega, \nu) : \nu = \omega$ except for value of x (any $\mathbb{R}$)$\}$

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Axioms

$\langle{:}*\rangle \ \langle x := *\rangle P \leftrightarrow$

$[{:}*] \ [x := *]P \leftrightarrow$

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Axioms

$\langle :* \rangle \ \langle x := * \rangle P \leftrightarrow \exists x\, P$

$[:*] \ [x := *] P \leftrightarrow \forall x\, P$

Nondeterministic Assignments & Ghosts of Choice

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---

② Or derived definition

Nondeterministic assignment $x := *$ not in HP syntax. ① Modular add

Syntax $\alpha ::= \ldots \mid x := *$

Semantics $[\![x := *]\!] = \{(\omega, \nu) : \nu = \omega$ except for value of x (any $\mathbb{R}$)$\}$

Axioms

$\langle{:}*\rangle \quad \langle x := * \rangle P \leftrightarrow \exists x\, P$

$[{:}*] \quad [x := *]P \leftrightarrow \forall x\, P$

② Or derived definition

Derived $x := * \stackrel{\text{def}}{\equiv}$

Nondeterministic Assignments & Ghosts of Choice

Nondeterministic assignment $x := *$ not in HP syntax. ① Modular add

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Axioms

$\langle{:}*\rangle \quad \langle x := *\rangle P \leftrightarrow \exists x\, P$

$[{:}*] \quad [x := *]P \leftrightarrow \forall x\, P$

② Or derived definition

Derived $x := * \stackrel{\text{def}}{\equiv} x' = 1 \cup x' = -1$

Nondeterministic assignment $x := *$ not in HP syntax. ① Modular add

Syntax: $\alpha ::= \ldots \mid x := *$

Semantics: $[\![x := *]\!] = \{(\omega, \nu) \ : \ \nu = \omega$ except for value of x (any $\mathbb{R}$)$\}$

Axioms:

$\langle :* \rangle$ $\langle x := * \rangle P \leftrightarrow \exists x\, P$

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Derived: $x := * \stackrel{\text{def}}{\equiv}$

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Semantics $[\![x := *]\!] = \{(\omega, \nu) : \nu = \omega \text{ except for value of } x \text{ (any } \mathbb{R})\}$

Axioms

$\langle :* \rangle \ \langle x := * \rangle P \leftrightarrow \exists x\, P$

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② Or derived definition

Derived $x := * \stackrel{\text{def}}{\equiv} x' = 1 \cup x' = -1$

Derived $x := * \stackrel{\text{def}}{\equiv} x' = 1; x' = -1$

Nondeterministic assignment $x := *$ not in HP syntax. ① Modular add

Syntax $\alpha ::= \ldots \mid x := *$

Semantics $[\![x := *]\!] = \{(\omega, \nu) : \nu = \omega \text{ except for value of } x \text{ (any } \mathbb{R})\}$

Axioms
$\langle:*\rangle\ \langle x := *\rangle P \leftrightarrow \exists x\, P$
$[:*]\ [x := *]P \leftrightarrow \forall x\, P$

discrete time?

continuous time?

② Or derived definition

Derived $x := * \overset{\text{def}}{\equiv} x' = 1 \cup x' = -1$

Derived $x := * \overset{\text{def}}{\equiv} x' = 1; x' = -1$

Nondeterministic assignment $x := *$ not in HP syntax. ① Modular add

Syntax	$\alpha ::= \dots \mid x := *$
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Axioms	$\langle :* \rangle\ \langle x := * \rangle P \leftrightarrow \exists x\, P$
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invisible time! time is relative.

② Or derived definition

Derived	$x := * \overset{\text{def}}{\equiv} x' = 1 \cup x' = -1$
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Nondeterministic assignment $x := *$ not in HP syntax. ① Modular add

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Axioms

$\langle{:}*\rangle \; \langle x := * \rangle P \leftrightarrow \exists x\, P$

$[{:}*] \; [x := *]P \leftrightarrow \forall x\, P$

invisible time! time is relative.

② Or derived definition

Derived $x := * \overset{\text{def}}{\equiv} x' = 1 \cup x' = -1$

Derived $x := * \overset{\text{def}}{\equiv} x' = 1; x' = -1$

$x := * \not\equiv x' = 1, t' = 1 \cup x' = -1, t' = 1$ visible time

Nondeterministic Assignments & Ghosts of Choice

Nondeterministic assignment $x := *$ not in HP syntax. ① Modular add

Syntax $\alpha ::= \ldots \mid x := *$

Semantics $[\![x := *]\!] = \{(\omega, \nu) : \nu = \omega \text{ except for value of } x \text{ (any } \mathbb{R})\}$

Axioms

$\langle{:}*\rangle \quad \langle x := * \rangle P \leftrightarrow \exists x\, P$

$[{:}*] \quad [x := *]P \leftrightarrow \forall x\, P$

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② Or derived definition

Derived $x := * \stackrel{\text{def}}{\equiv} x' = 1 \cup x' = -1$

Derived $x := * \stackrel{\text{def}}{\equiv} x' = 1; x' = -1$

I'm just a ghost of your imagination. I'm definable.

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4. $e/k \overset{\text{def}}{=}$ depends $\quad q = \frac{b}{c} \overset{\text{def}}{\equiv} qc = b$ where $c \neq 0$

Differential-Algebraic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e \cdot k \mid (e)'$
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Syntax	$P ::= e \geq k \mid p(e) \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

4. $e/k \stackrel{\text{def}}{=}$ depends $\quad q = \frac{b}{c} \stackrel{\text{def}}{\equiv} qc = b$ where $c \neq 0$

$$\{x' = \frac{2x}{c} \,\&\, c \neq 0 \wedge \frac{x+1}{c} > 0\}$$

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e \cdot k \mid (e)'$
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$$\{x' = \frac{2x}{c} \,\&\, c \neq 0 \wedge \frac{x+1}{c} > 0\} \rightsquigarrow q := *; ?qc = 1; \{x' = 2xq \,\&\, c \neq 0 \wedge (x+1)q > 0\}$$

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha\cup\beta \mid \alpha;\beta \mid \alpha^*$
Syntax	$P ::= e\geq k \mid p(e) \mid \neg P \mid P\wedge Q \mid \forall x\, P \mid \exists x\, Q \mid [\alpha]P \mid \langle\alpha\rangle P$

4. $e/k \stackrel{\text{def}}{=}$ depends $\quad q = \frac{b}{c} \stackrel{\text{def}}{\equiv} qc = b$ where $c \neq 0$

$$\{x'=\frac{2x}{c}\,\&\,c\neq 0 \wedge \frac{x+1}{c}>0\} \rightsquigarrow q:=*;?qc=1;\{x'=2xq\,\&\,c\neq 0\wedge (x+1)q>0\}$$

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inverse only of initial x

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
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change rate of q:

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e\cdot k \mid (e)'$
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change rate of q: $\quad q' = \left(\frac{1}{2x}\right)' = \frac{-2x'}{4x^2} = \frac{-2\frac{c}{2x}}{4x^2} = -\frac{c}{4x^3}$

Differential-Algebraic Ghosts

Syntax	$e ::= x \mid x' \mid f(e) \mid e+k \mid e \cdot k \mid (e)'$
Syntax	$\alpha ::= x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$
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still $1/x$

change rate of q: $\quad q' = \left(\frac{1}{2x}\right)' = \frac{-2x'}{4x^2} = \frac{-2\frac{c}{2x}}{4x^2} = -\frac{c}{4x^3}$

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$\{x'=\frac{c}{2x} \,\&\, 2$ … $x\neq0 \wedge cq>0\}$

continuously changing nondeterministic value

change rate of q: $\quad q' = \left(\frac{1}{2x}\right)' = \frac{-2x'}{4x^2} = \frac{-2\frac{c}{2x}}{4x^2} = -\frac{c}{4x^3}$

$$\rightsquigarrow \{x'=cq, q' = * \,\&\, q2x=1 \wedge 2x\neq0 \wedge cq>0\}$$

Differential-Algebraic Ghosts

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$$\{x'=\frac{c}{2x}\,\&\, \ldots \neq 0\wedge cq{>}0\}$$

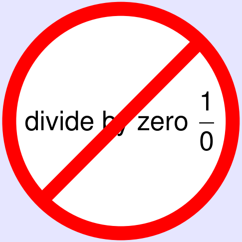
differential-algebraic ghost: auxiliary for the model

change rate of q: $\quad q' = \left(\frac{1}{2x}\right)' = \frac{-2x'}{4x^2} = \frac{-2\frac{c}{2x}}{4x^2} = -\frac{c}{4x^3}$

$$\rightsquigarrow \{x'{=}cq, q' = *\,\&\,q2x{=}1\wedge 2x{\neq}0\wedge cq{>}0\}$$

Divisions by Zero

Divisions



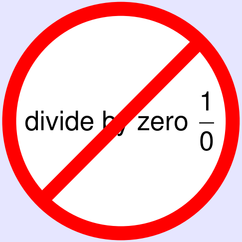
1. Scrutinize every division or possible singularity.
2. Missing requirements in the system.
3. Stopping distance $\frac{v^2}{2b}$ from initial velocity v

Don't divide by zero. It's not worth it.

~~Divide & Conquer~~ Divide & Regret

Divisions by Zero

Divisions



1. Scrutinize every division or possible singularity.
2. Missing requirements in the system.
3. Stopping distance $\frac{v^2}{2b}$ from initial velocity v
4. ... needs brakes to work $b \neq 0$ though ...

Don't divide by zero. It's not worth it.

~~Divide & Conquer~~ Divide & Regret

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