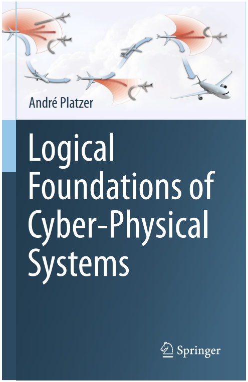
16: Winning & Proving Hybrid Games

Logical Foundations of Cyber-Physical Systems



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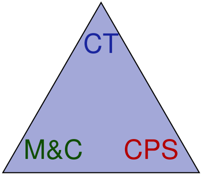
Computer Science Department

Outline

Outline

rigorous reasoning for adversarial dynamics
compositional reasoning from compositional semantics
modular addition of adversarial dynamics
axiomatization of dGL



analytical&semantical interaction
discrete+continuous+adversarial
fixpoints

CPS semantics
align semantics&reasoning
operational CPS effects

Outline

Differential Game Logic: Syntax

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^{d}$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game

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All Reals | Some Reals

Differential Game Logic: Syntax

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All Reals | Some Reals | Angel Wins

Differential Game Logic: Syntax

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"Angel has Wings $\langle \alpha \rangle$"

All Reals | Some Reals | Angel Wins | Demon Wins

Definition (Hybrid game α) $[\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathscr{S}) \to \wp(\mathscr{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathscr{S} \,:\, \omega_x^{\omega[\![e]\!]} \in X\}\\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathscr{S} : \varphi(r) \in X \text{ for some } \varphi{:}[0,r]{\to}\mathscr{S},\ \varphi \models x' = f(x)\}\\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X\\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X)\\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X))\\
\varsigma_{\alpha^*}(X) &= \bigcap\{Z \subseteq \mathscr{S} \,:\, X \cup \varsigma_\alpha(Z) \subseteq Z\}\\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

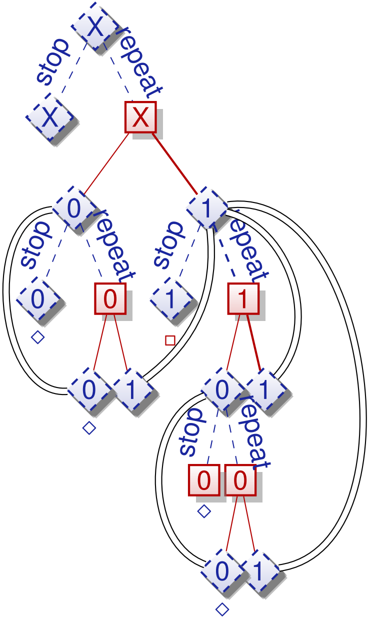
Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S})$

$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathscr{S} \,:\, \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\}\\
[\![\neg P]\!] &= ([\![P]\!])^\complement\\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!]\\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!])\\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

Differential Game Logic: Axiomatization

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



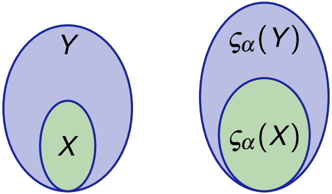
Consistency & Determinacy & Monotonicity

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\vDash \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Lemma (Monotonicity)

$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ *and* $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ *for all* $X \subseteq Y$



Consistency & Determinacy & Monotonicity

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\vDash \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

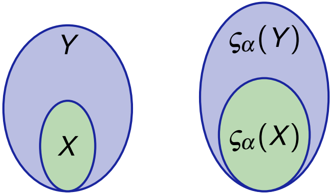
Corollary

Determined: At least one player wins: $\neg\langle\alpha\rangle\neg P \rightarrow [\alpha]P$ *so* $\langle\alpha\rangle\neg P \vee [\alpha]P$

Consistent: At most one player wins: $[\alpha]P \rightarrow \neg\langle\alpha\rangle\neg P$ *so* $\neg([\alpha]P \wedge \langle\alpha\rangle\neg P)$

Lemma (Monotonicity)

$$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y) \text{ and } \delta_\alpha(X) \subseteq \delta_\alpha(Y) \text{ for all } X \subseteq Y$$



Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\vDash \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Proof Sketch.

$\varsigma_{\alpha\cup\beta}(X^\complement)^\complement = (\varsigma_\alpha(X^\complement) \cup \varsigma_\beta(X^\complement))^\complement = \varsigma_\alpha(X^\complement)^\complement \cap \varsigma_\beta(X^\complement)^\complement = \delta_\alpha(X) \cap \delta_\beta(X) = \delta_{\alpha\cup\beta}(X)$ □

Lemma (Monotonicity)

$$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y) \textit{ and } \delta_\alpha(X) \subseteq \delta_\alpha(Y) \textit{ for all } X \subseteq Y$$

Proof Sketch.

- $X \subseteq Y$ so $X^\complement \supseteq Y^\complement$ so $\varsigma_\alpha(X^\complement) \supseteq \varsigma_\alpha(Y^\complement)$ so $\varsigma_{\alpha^d}(X) = (\varsigma_\alpha(X^\complement))^\complement \subseteq (\varsigma_\alpha(Y^\complement))^\complement = \varsigma_{\alpha^d}(Y)$.
- $\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\} \subseteq \varsigma_{\alpha^*}(Y) = \bigcap\{Z \subseteq \mathscr{S} : Y \cup \varsigma_\alpha(Z) \subseteq Z\}$ because $X \subseteq Y$ □

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\vDash \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Lemma (Monotonicity)

$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ *and* $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ *for all* $X \subseteq Y$

Theorem (Consistency & determinacy)

Hybrid games are consistent and determined, i.e., $\vDash \neg\langle\alpha\rangle\neg P \leftrightarrow [\alpha]P$.

Corollary (Axiom: Determinacy)

$[\cdot]\quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

Lemma (Monotonicity)

$$\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y) \text{ and } \delta_\alpha(X) \subseteq \delta_\alpha(Y) \text{ for all } X \subseteq Y$$

Corollary (Rule: Monotonicity)

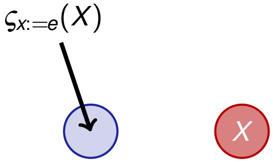
$$\text{M}\ \frac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q} \qquad \text{M}[\cdot]\ \frac{P \rightarrow Q}{[\alpha]P \rightarrow [\alpha]Q}$$

Outline

Hybrid Game Axioms

Axiom (Assignment) TOCL'15

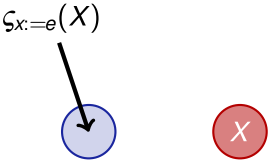
$\langle :=\rangle \quad \langle x := e\rangle p(x) \leftrightarrow$



Hybrid Game Axioms

Axiom (Assignment) TOCL'15

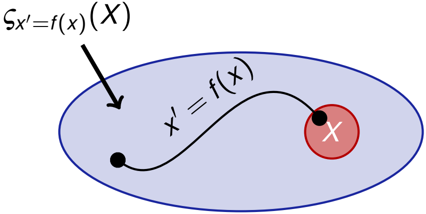
$\langle := \rangle \quad \langle x := e \rangle p(x) \leftrightarrow p(e)$



Hybrid Game Axioms

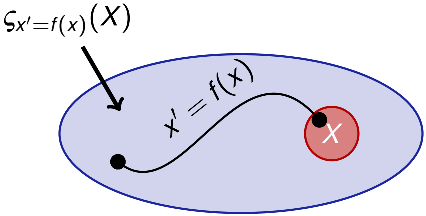
Axiom (Differential Equation) TOCL'15

$\langle ' \rangle \quad \langle x' = f(x) \rangle p(x) \leftrightarrow \qquad (y'(t) = f(y))$



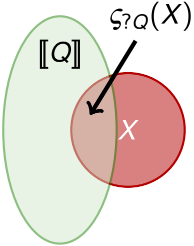
Axiom (Differential Equation) TOCL'15

$$\langle'\rangle \quad \langle x' = f(x)\rangle p(x) \leftrightarrow \exists t \geq 0 \, \langle x := y(t)\rangle p(x) \qquad (y'(t) = f(y))$$



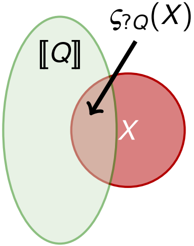
Axiom (Test / Challenge) TOCL'15

$\langle ? \rangle \quad \langle ?Q \rangle P \leftrightarrow$



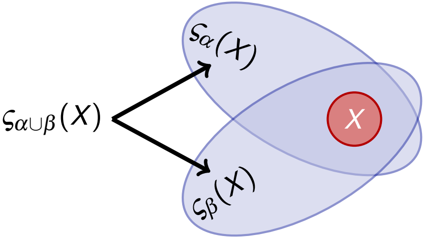
Axiom (Test / Challenge) TOCL'15

$\langle ? \rangle \quad \langle ?Q \rangle P \leftrightarrow Q \land P$



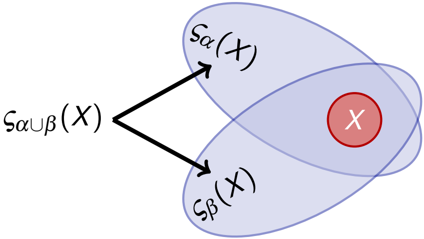
Axiom (Choice Game) TOCL'15

$\langle\cup\rangle \quad \langle\alpha \cup \beta\rangle P \leftrightarrow$



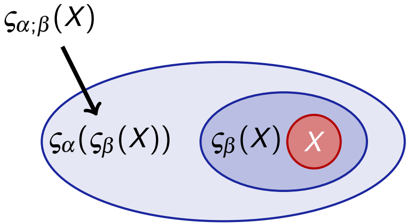
Axiom (Choice Game) TOCL'15

$\langle\cup\rangle \quad \langle\alpha\cup\beta\rangle P \leftrightarrow \langle\alpha\rangle P \vee \langle\beta\rangle P$



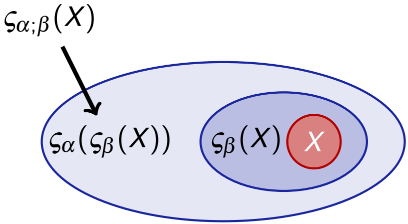
Axiom (Sequential Game) TOCL'15

$\langle;\rangle \quad \langle\alpha;\beta\rangle P \leftrightarrow$



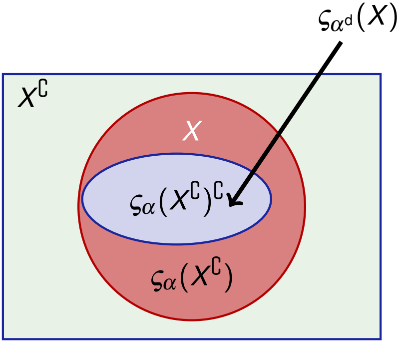
Axiom (Sequential Game) TOCL'15

$\langle;\rangle \quad \langle\alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$



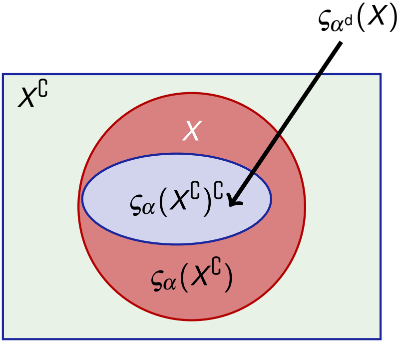
Axiom (Dual Game) — TOCL'15

$\langle^d\rangle \quad \langle \alpha^d \rangle P \leftrightarrow$



Axiom (Dual Game) TOCL'15

$\langle^d\rangle \quad \langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$



Example: Demon's Choice Derives by Duality

$$\frac{}{\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow}$$

Example: Demon's Choice Derives by Duality

$$\dfrac{}{\vdash \langle \alpha \cap \beta \rangle P \leftrightarrow \langle \alpha \rangle P \land \langle \beta \rangle P}$$

Example: Demon's Choice Derives by Duality

$$\alpha \cap \beta \equiv (\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}$$

$$\langle^d\rangle \frac{\overline{\vdash \langle(\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \wedge \langle\beta\rangle P}}{\vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \wedge \langle\beta\rangle P}$$

Example: Demon's Choice Derives by Duality

$\langle^d\rangle \quad \langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$$
\dfrac{\dfrac{\langle\cup\rangle\;\overline{\vdash \neg\langle\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}}\rangle\neg P \leftrightarrow \langle\alpha\rangle P \wedge \langle\beta\rangle P}}{\vdash \langle(\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \wedge \langle\beta\rangle P}\;\langle^d\rangle}{\vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \wedge \langle\beta\rangle P}
$$

Example: Demon's Choice Derives by Duality

$$\langle\cup\rangle \quad \langle\alpha\cup\beta\rangle P \leftrightarrow \langle\alpha\rangle P \lor \langle\beta\rangle P$$

$$\dfrac{\dfrac{\dfrac{\langle^d\rangle \; \overline{\vdash \neg(\langle\alpha^{\mathrm{d}}\rangle\neg P \lor \langle\beta^{\mathrm{d}}\rangle\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}}{\langle\cup\rangle \; \vdash \neg\langle\alpha^{\mathrm{d}}\cup\beta^{\mathrm{d}}\rangle\neg P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}}{\langle^d\rangle \; \vdash \langle(\alpha^{\mathrm{d}}\cup\beta^{\mathrm{d}})^{\mathrm{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}}{\vdash \langle\alpha\cap\beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}$$

Example: Demon's Choice Derives by Duality

$$\langle^d\rangle \quad \langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdash \neg(\neg\langle\alpha\rangle\neg\neg P \lor \neg\langle\beta\rangle\neg\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}{\vdash \neg(\langle\alpha^{\mathrm{d}}\rangle\neg P \lor \langle\beta^{\mathrm{d}}\rangle\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle^d\rangle}{\vdash \neg\langle\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}}\rangle\neg P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle\cup\rangle}{\vdash \langle(\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle^d\rangle}{\vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}
$$

Example: Demon's Choice Derives by Duality

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdash \langle\alpha\rangle P \land \langle\beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}{\vdash \neg(\neg\langle\alpha\rangle\neg\neg P \lor \neg\langle\beta\rangle\neg\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}}{\vdash \neg(\langle\alpha^{\mathrm{d}}\rangle\neg P \lor \langle\beta^{\mathrm{d}}\rangle\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle^{d}\rangle}{\vdash \neg\langle\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}}\rangle\neg P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle\cup\rangle}{\vdash \langle(\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle^{d}\rangle}{\vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}
$$

Example: Demon's Choice Derives by Duality

$$
\begin{array}{rl}
 & * \\
\hline
 & \vdash \langle\alpha\rangle P \land \langle\beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\hline
 & \vdash \neg(\neg\langle\alpha\rangle\neg\neg P \lor \neg\langle\beta\rangle\neg\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\hline
\langle^d\rangle & \vdash \neg(\langle\alpha^{\mathrm{d}}\rangle\neg P \lor \langle\beta^{\mathrm{d}}\rangle\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\hline
\langle\cup\rangle & \vdash \neg\langle\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}}\rangle\neg P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\hline
\langle^d\rangle & \vdash \langle(\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P \\
\hline
 & \vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P
\end{array}
$$

Example: Demon's Choice Derives by Duality

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash \langle\alpha\rangle P \land \langle\beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}}{\vdash \neg(\neg\langle\alpha\rangle\neg\neg P \lor \neg\langle\beta\rangle\neg\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}}{\vdash \neg(\langle\alpha^{\mathrm{d}}\rangle\neg P \lor \langle\beta^{\mathrm{d}}\rangle\neg P) \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle^d\rangle}{\vdash \neg\langle\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}}\rangle\neg P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle\cup\rangle}{\vdash \langle(\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}\langle^d\rangle}{\vdash \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P}$$

Derived axiom:

$$\langle\cap\rangle \quad \langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P$$

$$[\cdot]\ \dfrac{}{\vdash [\alpha \cap \beta]P \leftrightarrow}$$

Example: Demon's Choice Derives by Duality

$$[\cdot]\ \frac{}{\vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \vee [\beta]P}$$

Example: Demon's Choice Derives by Duality

$[\cdot]\quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$$\dfrac{\langle\cap\rangle \; \overline{\vdash \neg\langle\alpha\cap\beta\rangle\neg P \leftrightarrow [\alpha]P \vee [\beta]P}}{\vdash [\alpha\cap\beta]P \leftrightarrow [\alpha]P \vee [\beta]P}\,[\cdot]$$

Example: Demon's Choice Derives by Duality

$\langle\cap\rangle \quad \langle\alpha\cap\beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P$

$$\dfrac{\dfrac{}{\vdash \neg(\langle\alpha\rangle\neg P \land \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \lor [\beta]P}}{\langle\cap\rangle\ \dfrac{\vdash \neg\langle\alpha\cap\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}{[\cdot]\ \vdash [\alpha\cap\beta]P \leftrightarrow [\alpha]P \lor [\beta]P}}$$

Example: Demon's Choice Derives by Duality

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \neg\langle\alpha\rangle\neg P \lor \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}\,[\cdot]}{\vdash \neg(\langle\alpha\rangle\neg P \land \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash \neg\langle\alpha\cap\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}\,\langle\cap\rangle}{\vdash [\alpha\cap\beta]P \leftrightarrow [\alpha]P \lor [\beta]P}\,[\cdot]
$$

Example: Demon's Choice Derives by Duality

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash [\alpha]P \lor [\beta]P \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash \neg\langle\alpha\rangle\neg P \lor \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}\,[\cdot]}{\vdash \neg(\langle\alpha\rangle\neg P \land \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash \neg\langle\alpha\cap\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}\,\langle\cap\rangle}{\vdash [\alpha\cap\beta]P \leftrightarrow [\alpha]P \lor [\beta]P}\,[\cdot]
$$

Example: Demon's Choice Derives by Duality

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash [\alpha]P \lor [\beta]P \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash \neg\langle\alpha\rangle\neg P \lor \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}\,[\cdot]}{\vdash \neg(\langle\alpha\rangle\neg P \land \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash \neg\langle\alpha\cap\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}\,\langle\cap\rangle}{\vdash [\alpha\cap\beta]P \leftrightarrow [\alpha]P \lor [\beta]P}\,[\cdot]
$$

Example: Demon's Choice Derives by Duality

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash [\alpha]P \lor [\beta]P \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash \neg\langle\alpha\rangle\neg P \lor \neg\langle\beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}[\cdot]}{\vdash \neg(\langle\alpha\rangle\neg P \land \langle\beta\rangle\neg P) \leftrightarrow [\alpha]P \lor [\beta]P}}{\vdash \neg\langle\alpha \cap \beta\rangle\neg P \leftrightarrow [\alpha]P \lor [\beta]P}\langle\cap\rangle}{\vdash [\alpha \cap \beta]P \leftrightarrow [\alpha]P \lor [\beta]P}[\cdot]
$$

Derived axioms:

$[\cap]$ $[\alpha \cap \beta]P \leftrightarrow [\alpha]P \lor [\beta]P$

$\langle\cap\rangle$ $\langle\alpha \cap \beta\rangle P \leftrightarrow \langle\alpha\rangle P \land \langle\beta\rangle P$

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$\langle{:=}\rangle$ $\langle x := e\rangle p(x) \leftrightarrow p(e)$

$\langle{'}\rangle$ $\langle x' = f(x)\rangle p(x) \leftrightarrow \exists t{\geq}0\, \langle x := y(t)\rangle p(x)$

$\langle ? \rangle$ $\langle ?Q\rangle P \leftrightarrow Q \land P$

$\langle\cup\rangle$ $\langle\alpha \cup \beta\rangle P \leftrightarrow \langle\alpha\rangle P \lor \langle\beta\rangle P$

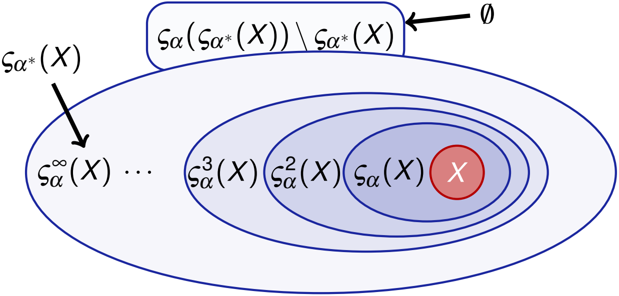
$\langle ; \rangle$ $\langle\alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$

$\langle^{d}\rangle$ $\langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$

Outline

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) = Z\}$$



Proofs for Loops

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$$

Proofs for Loops

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$ $\qquad$ $\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$

Proofs for Loops

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Lemma (Axiom:)

$\langle * \rangle$ $\quad \langle \alpha^* \rangle P \leftrightarrow$

Definition (Hybrid game α)

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Lemma (Axiom: Iteration)

$\langle{}^*\rangle \quad \langle\alpha^*\rangle P \leftrightarrow P \vee \langle\alpha\rangle\langle\alpha^*\rangle P$

Proofs for Loops

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$ $\qquad$ $\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$

Lemma (Axiom: Iteration)

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Lemma (Rule:)

$$FP \quad \frac{}{\langle\alpha^*\rangle P \rightarrow Q}$$

Proofs for Loops

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$ $\qquad$ $\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$

Lemma (Axiom: Iteration)

$\langle{}^*\rangle \quad \langle\alpha^*\rangle P \leftrightarrow P \vee \langle\alpha\rangle\langle\alpha^*\rangle P$

Lemma (Rule: Least Fixpoint)

$$\text{FP} \quad \frac{P \vee \langle\alpha\rangle Q \to Q}{\langle\alpha^*\rangle P \to Q}$$

Proofs for Loops

Definition (Hybrid game α)

$$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\} \qquad \varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$$

Lemma (Axiom: Iteration)

$$\langle * \rangle \quad \langle \alpha^* \rangle P \leftrightarrow P \vee \langle \alpha \rangle \langle \alpha^* \rangle P$$

Lemma (Rule: Least Fixpoint)

$$FP \quad \frac{P \vee \langle \alpha \rangle Q \rightarrow Q}{\langle \alpha^* \rangle P \rightarrow Q}$$

Corollary (Derived Rule:)

$$loop \quad \frac{}{P \rightarrow [\alpha^*]P}$$

Proofs for Loops

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$ $\qquad$ $\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$

Lemma (Axiom: Iteration)

$\langle^*\rangle \quad \langle\alpha^*\rangle P \leftrightarrow P \vee \langle\alpha\rangle\langle\alpha^*\rangle P$

Lemma (Rule: Least Fixpoint)

$$\text{FP} \quad \frac{P \vee \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$$

Corollary (Derived Rule: Loop Invariant)

$$\text{loop} \quad \frac{P \rightarrow [\alpha]P}{P \rightarrow [\alpha^*]P}$$

Proofs for Loops

Definition (Hybrid game α)

$\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$ $\qquad$ $\varsigma_{\alpha^*}(X) = X \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X))$

Lemma (Axiom: Iteration)

$\langle * \rangle \quad \langle \alpha^* \rangle P \leftrightarrow P \lor \langle \alpha \rangle \langle \alpha^* \rangle P$

Lemma (Rule: Least Fixpoint)

$$\text{FP}\ \frac{P \lor \langle \alpha \rangle Q \to Q}{\langle \alpha^* \rangle P \to Q}$$

Corollary (Derived Rule: Loop Invariant)

$$\text{loop}\ \frac{P \to [\alpha]P}{P \to [\alpha^*]P}$$

Proof

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdash P \to [\alpha]P}{\vdash P \to P \land [\alpha]P}}{\vdash P \to P \land \neg\langle \alpha \rangle \neg P}\,[\cdot]}{\vdash \neg P \lor \langle \alpha \rangle \neg P \to \neg P}}{\vdash \langle \alpha^* \rangle \neg P \to \neg P}\,\text{FP}}{\vdash P \to \neg\langle \alpha^* \rangle \neg P}}{\vdash P \to [\alpha^*]P}\,[\cdot]$$

Differential Game Logic: Axiomatization

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$\langle:=\rangle$ $\langle x:=e\rangle p(x) \leftrightarrow p(e)$

$\langle'\rangle$ $\langle x'=f(x)\rangle p(x) \leftrightarrow \exists t{\geq}0\,\langle x:=y(t)\rangle p(x)$

$\langle ? \rangle$ $\langle ?Q\rangle P \leftrightarrow Q \land P$

$\langle\cup\rangle$ $\langle\alpha\cup\beta\rangle P \leftrightarrow \langle\alpha\rangle P \lor \langle\beta\rangle P$

$\langle;\rangle$ $\langle\alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$

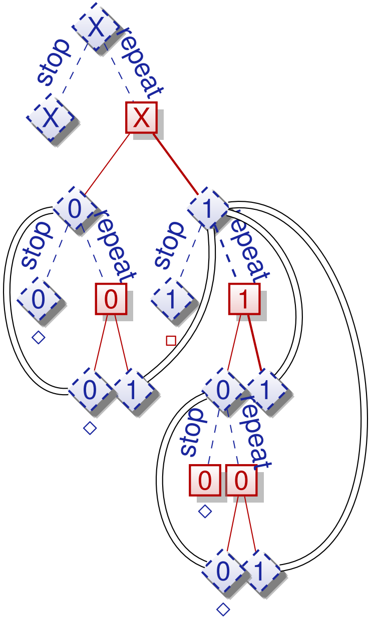
$\langle^*\rangle$ $\langle\alpha^*\rangle P \leftrightarrow P \lor \langle\alpha\rangle\langle\alpha^*\rangle P$

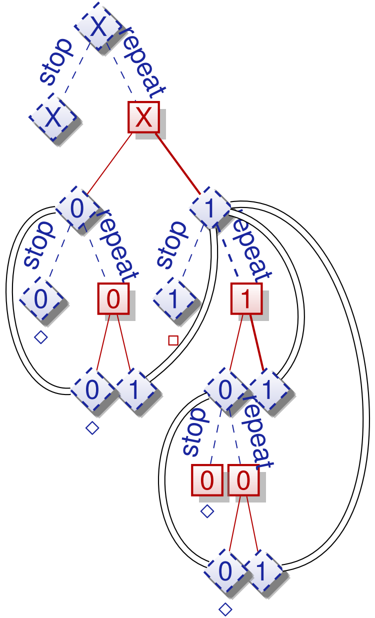
$\langle^d\rangle$ $\langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$

M $\dfrac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q}$

FP $\dfrac{P \lor \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$

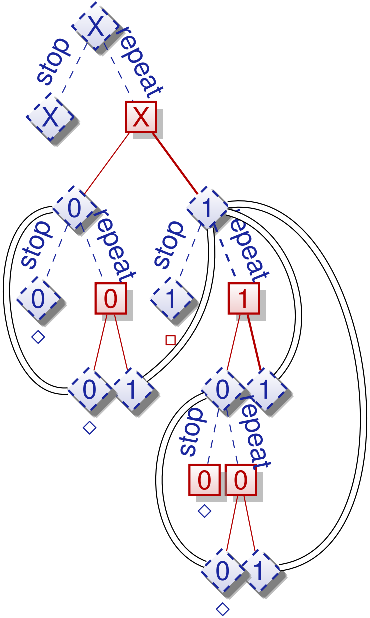
$$\langle^d\rangle \frac{}{x = 0 \vdash \langle (x := 0 \cup x := 1)^\times \rangle x = 0}$$





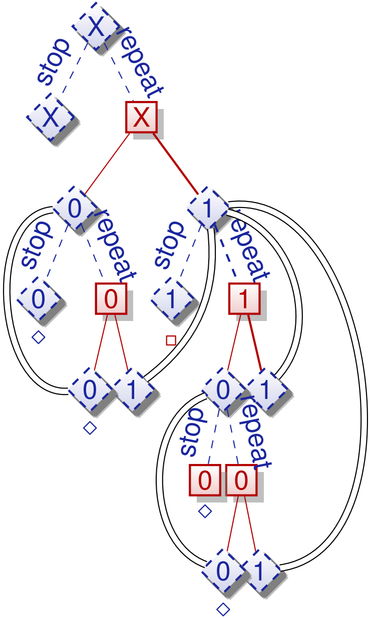
$$\text{ind}\ \dfrac{\dfrac{}{x=0 \vdash [(x:=0 \cap x:=1)^*]x=0}}{x=0 \vdash \langle (x:=0 \cup x:=1)^\times \rangle x=0}\ \langle^d\rangle$$

$$\dfrac{\dfrac{\dfrac{}{x=0 \vdash [x:=0 \cap x:=1]x=0}[\cdot]}{x=0 \vdash [(x:=0 \cap x:=1)^*]x=0}\text{ind}}{x=0 \vdash \langle (x:=0 \cup x:=1)^\times \rangle x=0}\langle^d\rangle$$



Example Proof: Dual Filibuster

$$
\begin{array}{rl}
\langle^d\rangle & \dfrac{}{x=0 \vdash \neg\langle x:=0 \cap x:=1\rangle \neg x=0} \\
[\cdot] & \dfrac{}{x=0 \vdash [x:=0 \cap x:=1]x=0} \\
\text{ind} & \dfrac{}{x=0 \vdash [(x:=0 \cap x:=1)^*]x=0} \\
\langle^d\rangle & \dfrac{}{x=0 \vdash \langle (x:=0 \cup x:=1)^\times \rangle x=0}
\end{array}
$$



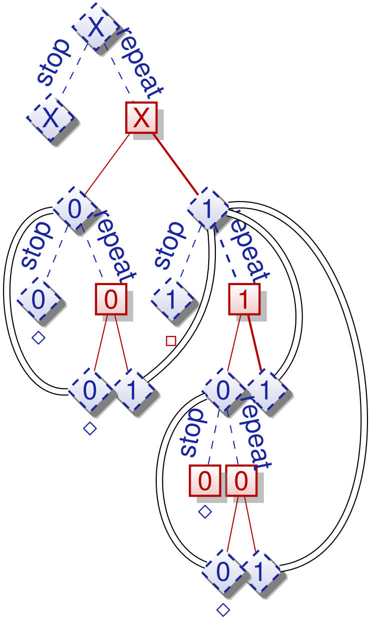
$\langle\cup\rangle$ $x = 0 \vdash \langle x := 0 \cup x := 1\rangle x = 0$

$\langle^d\rangle$ $x = 0 \vdash \neg\langle x := 0 \cap x := 1\rangle \neg x = 0$

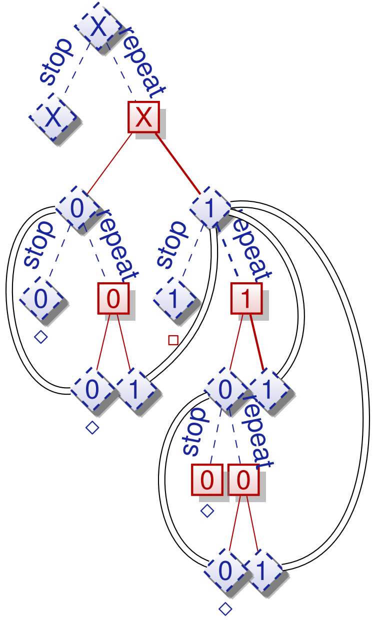
$[\cdot]$ $x = 0 \vdash [x := 0 \cap x := 1] x = 0$

ind $x = 0 \vdash [(x := 0 \cap x := 1)^*] x = 0$

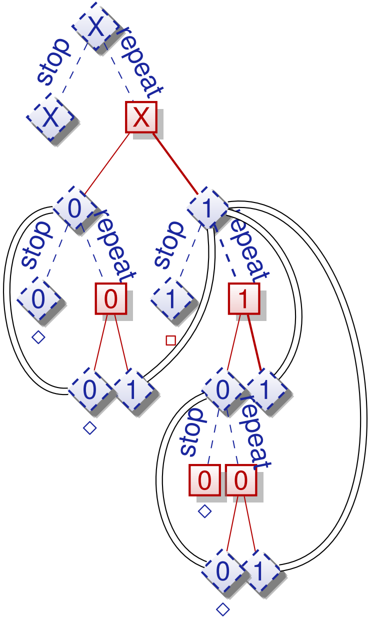
$\langle^d\rangle$ $x = 0 \vdash \langle (x := 0 \cup x := 1)^{\times}\rangle x = 0$



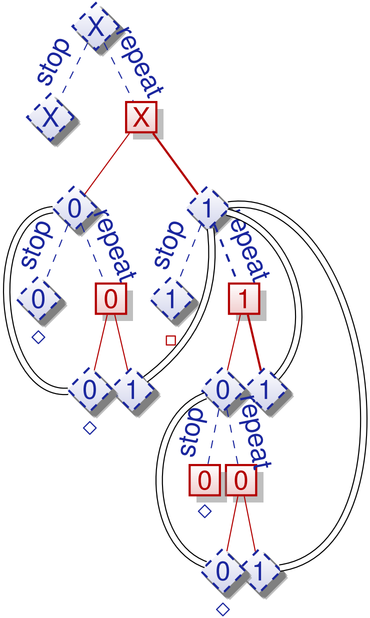
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x=0 \vdash \langle x:=0\rangle x=0 \lor \langle x:=1\rangle x=0}\langle:=\rangle}{x=0 \vdash \langle x:=0 \cup x:=1\rangle x=0}\langle\cup\rangle}{x=0 \vdash \neg\langle x:=0 \cap x:=1\rangle \neg x=0}\langle^d\rangle}{x=0 \vdash [x:=0 \cap x:=1]x=0}[\cdot]}{x=0 \vdash [(x:=0 \cap x:=1)^*]x=0}\text{ind}}{x=0 \vdash \langle (x:=0 \cup x:=1)^\times\rangle x=0}\langle^d\rangle$$



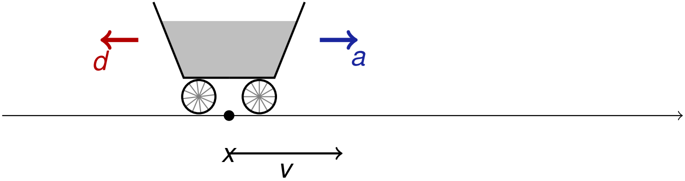
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{x=0 \vdash 0=0 \lor 1=0}\ \mathbb{R}}{x=0 \vdash \langle x:=0\rangle x=0 \lor \langle x:=1\rangle x=0}\ \langle:=\rangle}{x=0 \vdash \langle x:=0 \cup x:=1\rangle x=0}\ \langle\cup\rangle}{x=0 \vdash \neg\langle x:=0 \cap x:=1\rangle\neg x=0}\ \langle^d\rangle}{x=0 \vdash [x:=0 \cap x:=1]x=0}\ [\cdot]}{x=0 \vdash [(x:=0 \cap x:=1)^*]x=0}\ \text{ind}}{x=0 \vdash \langle (x:=0 \cup x:=1)^\times\rangle x=0}\ \langle^d\rangle$$



$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{x=0 \vdash 0=0 \lor 1=0}\ \mathbb{R}}{x=0 \vdash \langle x:=0\rangle x=0 \lor \langle x:=1\rangle x=0}\ \langle:=\rangle}{x=0 \vdash \langle x:=0 \cup x:=1\rangle x=0}\ \langle\cup\rangle}{x=0 \vdash \neg\langle x:=0 \cap x:=1\rangle \neg x=0}\ \langle^d\rangle}{x=0 \vdash [x:=0 \cap x:=1]x=0}\ [\cdot]}{x=0 \vdash [(x:=0 \cap x:=1)^*]x=0}\ \text{ind}}{x=0 \vdash \langle (x:=0 \cup x:=1)^{\times}\rangle x=0}\ \langle^d\rangle
$$



Example Proof: Push-around Cart



$$\text{ind}\quad \overline{J \vdash [\big((d := 1 \cap d := -1);(a := 1 \cup a := -1);\{x'=v, v'=a+d\}\big)^*]x \geq 0}$$

$$\dfrac{\dfrac{}{J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J}\,[;]}{J \vdash [\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*]x\geq 0}\,\text{ind}$$

Example Proof: Push-around Cart

$$
\dfrac{\dfrac{\overline{J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J}\;[\cap]}{J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J}\;[;]}{J \vdash [\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*]x \geq 0}\;\text{ind}
$$

Example Proof: Push-around Cart

$$
\dfrac{
 \dfrac{
  \dfrac{
   \dfrac{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J \lor [d:=-1]\ldots}
   {J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J}\ \lor\mathrm{R,WR}
  }
  {J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J}\ [\cap]
 }
 {J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J}\ [;]
}
{J \vdash [\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*]x\geq 0}\ \mathrm{ind}
$$

Example Proof: Push-around Cart

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}\,[:=]}{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \lor [d:=-1]\dots}\,\lor\mathrm{R,WR}}{J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}\,[\cap]}{J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}\,[;]}{J \vdash [\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*]x \geq 0}\,\mathrm{ind}
$$

Example Proof: Push-around Cart

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{J \vdash [(a:=1 \cup a:=-1);\{x'=v,v'=a+1\}]J}\;[;]}{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}\;[:=]}{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \lor [d:=-1]\dots}\;\lor R,WR}{J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}\;[\cap]}{J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}\;[;]}{J \vdash [\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*]x \geq 0}\;\text{ind}$$

Example Proof: Push-around Cart

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{J \vdash [a:=1 \cup a:=-1][\{x'=v, v'=a+1\}]J}[\cup]}{J \vdash [(a:=1 \cup a:=-1);\{x'=v, v'=a+1\}]J}[;]}{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J}[:=]}{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J \vee [d:=-1]\dots}\vee R,WR}{J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J}[\cap]}{J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}]J}[;]}{J \vdash [\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\big)^*]x \geq 0}\text{ind}
$$

Example Proof: Push-around Cart

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{J \vdash [a:=1][\{x'=v,v'=a+1\}]J \land [a:=-1][\{x'=v,v'=a+1\}]J}[:=]}{J \vdash [a:=1 \cup a:=-1][\{x'=v,v'=a+1\}]J}[\cup]}{J \vdash [(a:=1 \cup a:=-1);\{x'=v,v'=a+1\}]J}[;]}{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}[:=]}{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \lor [d:=-1]\ldots}\lor R,WR}{J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}[\cap]}{J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}[;]}{J \vdash [\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*]x \geq 0}\text{ind}
$$

$$
\begin{array}{ll}
 & J \vdash [\{x' = v, v' = 1+1\}]J \land [\{x' = v, v' = -1+1\}]J \\
\hline
[:=] & J \vdash [a:=1][\{x' = v, v' = a+1\}]J \land [a:=-1][\{x' = v, v' = a+1\}]J \\
\hline
[\cup] & J \vdash [a:=1 \cup a:=-1][\{x' = v, v' = a+1\}]J \\
\hline
[;] & J \vdash [(a:=1 \cup a:=-1); \{x' = v, v' = a+1\}]J \\
\hline
[:=] & J \vdash [d:=1][(a:=1 \cup a:=-1); \{x' = v, v' = a+d\}]J \\
\hline
\lor R, WR & J \vdash [d:=1][(a:=1 \cup a:=-1); \{x' = v, v' = a+d\}]J \lor [d:=-1]\ldots \\
\hline
[\cap] & J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1); \{x' = v, v' = a+d\}]J \\
\hline
[;] & J \vdash [(d:=1 \cap d:=-1); (a:=1 \cup a:=-1); \{x' = v, v' = a+d\}]J \\
\hline
\text{ind} & J \vdash [\big((d:=1 \cap d:=-1); (a:=1 \cup a:=-1); \{x'=v, v'=a+d\}\big)^*]x \geq 0
\end{array}
$$

Example Proof: Push-around Cart

$$
\begin{array}{rl}
 & J \vdash [\{x'=v,v'=1+1\}]J \land [\{x'=v,v'=-1+1\}]J \\
\hline
[:=] & J \vdash [a:=1][\{x'=v,v'=a+1\}]J \land [a:=-1][\{x'=v,v'=a+1\}]J \\
\hline
[\cup] & J \vdash [a:=1 \cup a:=-1][\{x'=v,v'=a+1\}]J \\
\hline
[;] & J \vdash [(a:=1 \cup a:=-1);\{x'=v,v'=a+1\}]J \\
\hline
[:=] & J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \\
\hline
\lor R,WR & J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \lor [d:=-1]\ldots \\
\hline
[\cap] & J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \\
\hline
[;] & J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \\
\hline
\text{ind} & J \vdash [\big((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*]x \geq 0
\end{array}
$$

$$J \stackrel{\text{def}}{\equiv} x \geq 0 \land v \geq 0$$

Example Proof: Push-around Cart

$$
\cfrac{
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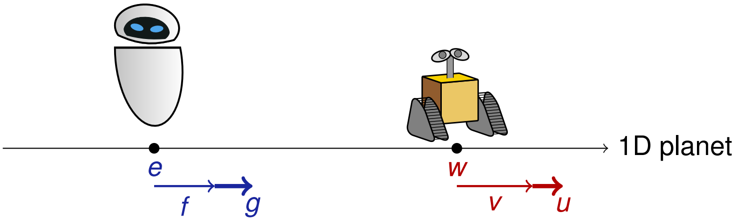
\cfrac{
\cfrac{
\cfrac{
\cfrac{
J \vdash [\{x'=v,v'=1+1\}]J \land [\{x'=v,v'=-1+1\}]J
}{J \vdash [a:=1][\{x'=v,v'=a+1\}]J \land [a:=-1][\{x'=v,v'=a+1\}]J}\,[:=]
}{J \vdash [a:=1 \cup a:=-1][\{x'=v,v'=a+1\}]J}\,[\cup]
}{J \vdash [(a:=1 \cup a:=-1);\{x'=v,v'=a+1\}]J}\,[;]
}{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}\,[:=]
}{J \vdash [d:=1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J \lor [d:=-1]\ldots}\,\lor R, WR
}{J \vdash [d:=1 \cap d:=-1][(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}\,[\cap]
}{J \vdash [(d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}]J}\,[;]
}{J \vdash [((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\})^*]x \geq 0}\,\text{ind}
$$

$$J \overset{\text{def}}{\equiv} x \geq 0 \land v \geq 0$$

$$
[{}'],[:=]\,\cfrac{x \geq 0 \land v \geq 0 \vdash \forall t \geq 0\,(x+vt+t^2 \geq 0 \land v+2t \geq 0)}{J \vdash [\{x'=v,v'=1+1\}]J}
$$

$$
[{}'],[:=]\,\cfrac{x \geq 0 \land v \geq 0 \vdash \forall t \geq 0\,(x+vt \geq 0 \land v \geq 0)}{J \vdash [\{x'=v,v'=0\}]J}
$$

Example Proof: WALL·E and EVE Robot Dance



$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \rightarrow \\
&\langle ((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\}^{d} \\
&)^{\times} \rangle \, (w-e)^2 \le 1
\end{aligned}
$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u

Outline

Differential Game Logic: Axiomatization

$$[\cdot] \quad [\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\langle:=\rangle \quad \langle x := e\rangle p(x) \leftrightarrow p(e)$$

$$\langle'\rangle \quad \langle x' = f(x)\rangle p(x) \leftrightarrow \exists t{\geq}0\, \langle x := y(t)\rangle p(x)$$

$$\langle ?\rangle \quad \langle ?Q\rangle P \leftrightarrow Q \wedge P$$

$$\langle\cup\rangle \quad \langle\alpha\cup\beta\rangle P \leftrightarrow \langle\alpha\rangle P \vee \langle\beta\rangle P$$

$$\langle;\rangle \quad \langle\alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$$

$$\langle^*\rangle \quad \langle\alpha^*\rangle P \leftrightarrow P \vee \langle\alpha\rangle\langle\alpha^*\rangle P$$

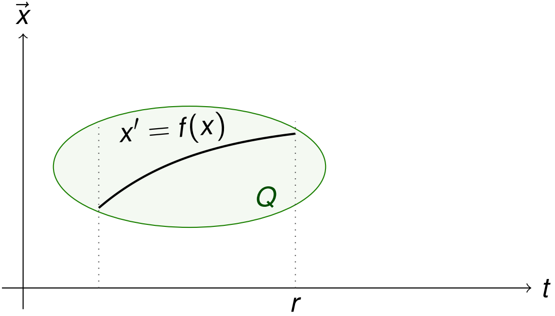
$$\langle^d\rangle \quad \langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$$

$$\text{M} \quad \frac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q}$$

$$\text{FP} \quad \frac{P \vee \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$$

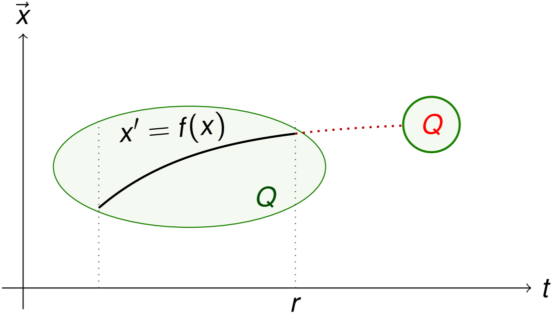
Defining Evolution Domain Constraints

$x' = f(x) \& Q$ $\qquad$ $x' = f(x); ?(Q)$

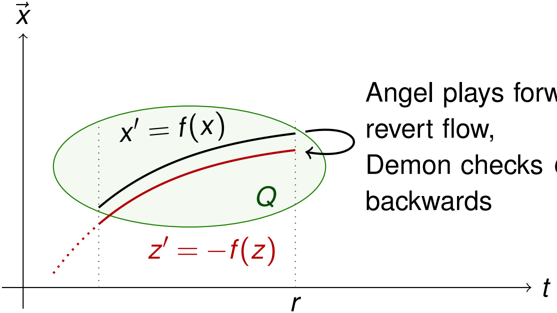


Defining Evolution Domain Constraints

$x' = f(x) \,\&\, Q$ $\qquad$ $x' = f(x); ?(Q)$



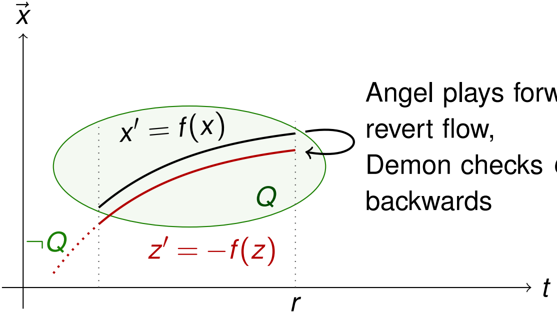
$x' = f(x) \,\&\, Q$ $\qquad$ $x' = f(x); (z := x; z' = -f(z))^{d}; ?(Q(z))$



Angel plays forward revert flow, Demon checks Q backwards

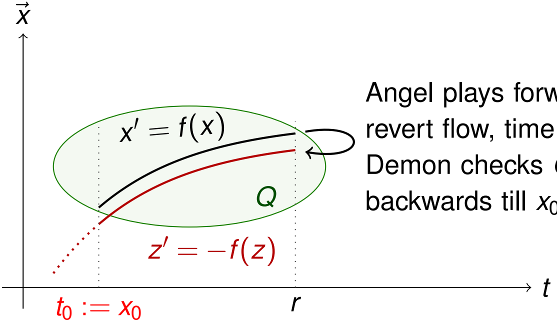
Defining Evolution Domain Constraints

$x' = f(x) \,\&\, Q$ $\qquad$ $x' = f(x); (z := x; z' = -f(z))^d; ?(Q(z))$



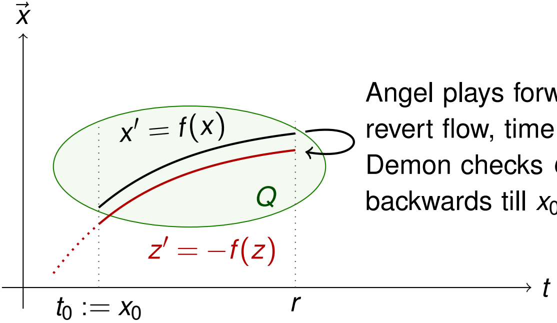
Angel plays forward revert flow, Demon checks Q backwards

$$x' = f(x) \,\&\, Q \equiv t_0 := x_0; x' = f(x); (z := x; z' = -f(z))^d; ?(z_0 \geq t_0 \rightarrow Q(z))$$



Angel plays forward
revert flow, time x_0;
Demon checks Q
backwards till x_0

$$x' = f(x) \,\&\, Q \equiv t_0 := x_0; x' = f(x); (z := x; z' = -f(z))^d; ?(z_0 \geq t_0 \to Q(z))$$



Angel plays forward
revert flow, time x_0;
Demon checks Q
backwards till x_0

Lemma

Evolution domains definable by games

Outline

Differential Game Logic: Denotational Semantics

Definition (Hybrid game α) $[\![\cdot]\!] : \mathrm{HG} \to (\wp(\mathscr{S}) \to \wp(\mathscr{S}))$

$$
\begin{aligned}
\varsigma_{x:=e}(X) &= \{\omega \in \mathscr{S} : \omega_x^{\omega[\![e]\!]} \in X\} \\
\varsigma_{x'=f(x)}(X) &= \{\varphi(0) \in \mathscr{S} : \varphi(r) \in X \text{ for some } \varphi{:}[0,r]\to\mathscr{S},\ \varphi \models x' = f(x)\} \\
\varsigma_{?Q}(X) &= [\![Q]\!] \cap X \\
\varsigma_{\alpha\cup\beta}(X) &= \varsigma_\alpha(X) \cup \varsigma_\beta(X) \\
\varsigma_{\alpha;\beta}(X) &= \varsigma_\alpha(\varsigma_\beta(X)) \\
\varsigma_{\alpha^*}(X) &= \bigcap\{Z \subseteq \mathscr{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\} \\
\varsigma_{\alpha^d}(X) &= (\varsigma_\alpha(X^\complement))^\complement
\end{aligned}
$$

Definition (dGL Formula P) $[\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S})$

$$
\begin{aligned}
[\![e_1 \geq e_2]\!] &= \{\omega \in \mathscr{S} : \omega[\![e_1]\!] \geq \omega[\![e_2]\!]\} \\
[\![\neg P]\!] &= ([\![P]\!])^\complement \\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\langle\alpha\rangle P]\!] &= \varsigma_\alpha([\![P]\!]) \\
[\![[\alpha]P]\!] &= \delta_\alpha([\![P]\!])
\end{aligned}
$$

Differential Game Logic: Axiomatization

$[\cdot]$ $[\alpha]P \leftrightarrow \neg\langle\alpha\rangle\neg P$

$\langle:=\rangle$ $\langle x := e\rangle p(x) \leftrightarrow p(e)$

$\langle'\rangle$ $\langle x' = f(x)\rangle p(x) \leftrightarrow \exists t{\geq}0\, \langle x := y(t)\rangle p(x)$

$\langle?\rangle$ $\langle ?Q\rangle P \leftrightarrow Q \wedge P$

$\langle\cup\rangle$ $\langle\alpha \cup \beta\rangle P \leftrightarrow \langle\alpha\rangle P \vee \langle\beta\rangle P$

$\langle;\rangle$ $\langle\alpha;\beta\rangle P \leftrightarrow \langle\alpha\rangle\langle\beta\rangle P$

$\langle^*\rangle$ $\langle\alpha^*\rangle P \leftrightarrow P \vee \langle\alpha\rangle\langle\alpha^*\rangle P$

$\langle^d\rangle$ $\langle\alpha^{\mathrm{d}}\rangle P \leftrightarrow \neg\langle\alpha\rangle\neg P$

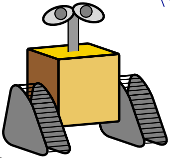
M $\dfrac{P \rightarrow Q}{\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q}$

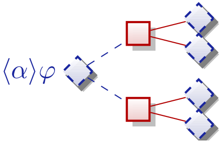
FP $\dfrac{P \vee \langle\alpha\rangle Q \rightarrow Q}{\langle\alpha^*\rangle P \rightarrow Q}$

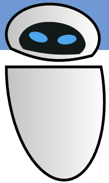
Summary

differential game logic

dGL = GL + HG = dL + ^{d}



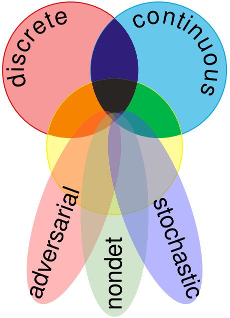




- Axiomatics for hybrid games
- Proving winning strategies

Next chapter

1. Soundness
2. Proofs
3. Separations



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